

A p -ADIC EISENSTEIN MEASURE FOR VECTOR-WEIGHT AUTOMORPHIC FORMS

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ABSTRACT. We construct a p -adic Eisenstein measure with values in the space of vector-weight p -adic automorphic forms on certain unitary groups. This measure allows us to p -adically interpolate special values of certain vector-weight C^∞ -automorphic forms, including Eisenstein series, as their weights vary.

We also explain how to extend our methods to the case of Siegel modular forms and how to recover Nicholas Katz's p -adic families of Eisenstein series for Hilbert modular forms.

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1. INTRODUCTION

The significance of p -adic families of Eisenstein series as a tool in number theory (especially for the construction of p -adic L -functions) is well-established, for example in the work of Pierre Deligne, Nicholas Katz, Kenneth Ribet, and Jean-Pierre Serre [Ser73, DR80, Kat78]. In [Eis11], we constructed a p -adic Eisenstein measure for (scalar-weight) automorphic forms on unitary groups of signature (n, n) . (By an *Eisenstein measure*, we mean a p -adic measure valued in a space of p -adic automorphic forms and whose values at locally constant functions are Eisenstein series.)

Each of the constructions mentioned above concerns only automorphic forms of scalar weight. Automorphic forms on groups of rank 1 (e.g. modular forms and Hilbert modular forms) can only have scalar weights. Automorphic forms on groups of higher rank, however, need not have scalar weights. In order to complete a construction of p -adic L -functions for automorphic forms on unitary groups in full generality as in [EHLS], one needs a p -adic Eisenstein measure that takes values in the space of p -adic vector- (not necessarily scalar-) weight automorphic forms. That is, one must work with automorphic forms whose weights are representations of dimension greater than 1. (By a vector-weight automorphic form, we mean an automorphic form whose weight is a representation whose highest weight $\lambda_n \geq \dots \geq \lambda_1$ is not required to have $\lambda_i = \lambda_{i+1}$ for all i , i.e. an automorphic form whose weight is not required to be a one-dimensional representation.)

The main result of this paper is the construction in Section 5 of a p -adic measure that takes values in the space of (not necessarily scalar-weight) automorphic forms on unitary groups of signature (n, n) . In particular, Theorem 15 gives a p -adic Eisenstein measure with values in the space of vector-weight automorphic forms. As explained in Theorem 16, this measure together with the results of Section 4 allows us to p -adically interpolate the values of certain vector-weight C^∞ -automorphic forms, including Eisenstein series, as the (highest) weights of these automorphic forms vary. Note that this is the first ever construction of a p -adic Eisenstein measure taking values in the space of *vector*-weight automorphic forms on unitary groups.

We also explain in Section 6 how to generalize the results of Section 5 to the case of Siegel modular forms, i.e. automorphic forms on symplectic groups. In that setting, in the case where $n = 1$, we are in exactly the situation of [Kat78], in which Katz constructs a p -adic Eisenstein measure for Hilbert modular forms. As demonstrated in Section 6.1, the setup in the earlier sections of the paper makes the connection between our Eisenstein measure and the Eisenstein measure in [Kat78] almost transparent (more so than in [Eis11]).

The main anticipated application is to the construction of p -adic L -functions for unitary groups (e.g. [EHLS]). We also note that in [Eis], we will use this measure to construct measures that take values in the spaces of p -adic automorphic forms on unitary groups of arbitrary signature.

1.1. Overview and structure of the paper. In Section 2, we introduce the conventions with which we will work, as well as standard background results necessary for this paper. The conventions and background are similar to those in [Eis12]. The background is quite technical; we have summarized just what is needed for this paper. The reader can find substantial reference materials on the background; for the reader seeking details on the background material, we recommend [Shi97, Shi00] for the theory of C^∞ -automorphic forms and Eisenstein series on unitary groups, [Lan12, Lan08] for the algebraic geometric background and a discussion of algebraically defined q -expansions, and [Hid04, Hid05] for the theory of p -adic automorphic forms.

In Section 3, we define certain scalar-weight Eisenstein series and automorphic forms on unitary groups of signature (n, n) . Note this set of automorphic forms includes the Eisenstein series defined in [Eis11, Section 2] but also includes other automorphic forms as well. We need this larger space of automorphic forms in order to construct a p -adic measure with values in the space of vector-weight automorphic forms in Section 5, whereas in [Eis11], we only were concerned with p -adic families of scalar-weight automorphic forms. Like in [Eis11], we work adelically. The format of this section and the formulation of the main result of this section (Theorem 2) is closer to that of [Kat78], though, so that the reader can see parallels with the analogous construction in [Kat78] (which is useful in Section 6.1 when we compare our results to those obtained in the setting of [Kat78]).

Section 4 discusses differential operators that are necessary for comparing the values of certain C^∞ -automorphic forms and certain p -adic-automorphic forms. These differential operators are closely related to the differential operators discussed in [Eis12]. Note that because we work with vector-weight automorphic forms, and not just scalar-weight automorphic forms, in this paper, we need more differential operators than we did in [Eis11], which handled only the case of scalar-weight automorphic forms.

Section 5, as noted above, contains the main results of the paper, namely the construction of a p -adic Eisenstein measure and the p -adic interpolation of special values of certain automorphic forms. The format of Section 5 closely parallels the construction of a p -adic Eisenstein measure in [Kat78, Sections 3.4 and 4.2]. We also explain in Remark 17 precisely how the Eisenstein measure of [Eis11] and the Eisenstein measure given in Theorem 15 are related. Note that for $n \geq 2$, the measure in Theorem 15 is on a larger group than the the measure in [Eis11]. In order to construct a measure with values in the space of *vector*-weight automorphic forms without fixing a partition of n , this larger group is necessary. (In [Eis11], we had implied that fixing a partition of n is necessary, but it turns out that with this larger group, we do not need to fix a partition of n and can consider a larger class of automorphic forms all at once.) We also note that the Eisenstein measures in [Eis] all use this measure as a starting point.

In Section 6, as mentioned above, we comment on how to extend the results of this paper to the case of Siegel modular forms, i.e. automorphic forms on symplectic groups. The fact that our presentation in Section 5 closely follows the approach in [Kat78, Sections 3.4 and 4.2] also allows us to recover the Eisenstein measure of [Kat78] with ease in Section 6.1.

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2. CONVENTIONS AND BACKGROUND

In Section 2.1, we introduce the conventions that we will use throughout the paper. In Section 2.2, we briefly summarize the necessary background on automorphic forms on unitary groups. For a more detailed discussion of automorphic forms on unitary groups, we recommend [Shi97, Shi00, Lan08, Hid04, Eis12, EHLS]; for a thorough discussion in the analogous case of Hilbert modular forms, the reader should consult [Kat78, Section 1].

2.1. Conventions. Once and for all, fix a CM field K with maximal totally real subfield E . Fix a prime p that is unramified in K and such that each prime of E dividing p splits completely in K . Fix embeddings

$$\begin{aligned}\iota_\infty : \bar{\mathbb{Q}} &\hookrightarrow \mathbb{C} \\ \iota_p : \bar{\mathbb{Q}} &\hookrightarrow \bar{\mathbb{C}}_p,\end{aligned}$$

and fix an isomorphism

$$\iota : \bar{\mathbb{C}}_p \xrightarrow{\sim} \mathbb{C}$$

satisfying $\iota \circ \iota_p = \iota_\infty$. From here on, we identify $\bar{\mathbb{Q}}$ with $\iota_p(\bar{\mathbb{Q}})$ and $\iota_\infty(\bar{\mathbb{Q}})$. Let $\mathcal{O}_{\mathbb{C}_p}$ denote the ring of integers in \mathbb{C}_p .

Fix a CM type Σ for K/\mathbb{Q} . For each element $\sigma \in \text{Hom}(E, \bar{\mathbb{Q}})$, we also write σ to denote the unique element of Σ prolonging $\sigma : E \hookrightarrow \bar{\mathbb{Q}}$ (when no confusion can arise). For each element $x \in K$, denote by \bar{x} the image of x under the unique non-trivial element $\epsilon \in \text{Gal}(K/E)$, and let $\bar{\sigma} = \sigma \circ \epsilon$.

Given an element a of E , we identify it with an element of $E \otimes \mathbb{R}$ via the embedding

$$\begin{aligned}(1) \quad & E \hookrightarrow E \otimes \mathbb{R} \\ (2) \quad & a \mapsto (\sigma(a))_{\sigma \in \Sigma}.\end{aligned}$$

We identify $a \in K$ with an element of $K \otimes \mathbb{C} \xrightarrow{\sim} (E \otimes \mathbb{C}) \times (E \otimes \mathbb{C})$ via the embedding

$$\begin{aligned}K &\hookrightarrow K \otimes \mathbb{C} \\ a &\mapsto ((\sigma(a))_{\sigma \in \Sigma}, (\bar{\sigma}(a))_{\sigma \in \Sigma}).\end{aligned}$$

Let $d = (d_v)_{v \in \Sigma} \in \mathbb{Z}^\Sigma$, and let $a = (a_v)_{v \in \Sigma}$ be an element of \mathbb{C}^Σ or \mathbb{C}_p^Σ . We denote by a^d the element of \mathbb{C} or \mathbb{C}_p defined by

$$a^d := \prod_{v \in \Sigma} a_v^{d_v}.$$

If $e = (e_v)_{v \in \Sigma} \in \mathbb{Z}^\Sigma$, we denote by $d + e$ the tuple defined by

$$d + e = (d_v + e_v)_{v \in \Sigma} \in \mathbb{Z}^\Sigma.$$

If $k \in \mathbb{Z}$, we denote by $k + d$ or $d + k$ the element

$$k + d = d + k = (d_v + k)_{v \in \Sigma} \in \mathbb{Z}^\Sigma.$$

For any ring R , we denote the ring of $n \times n$ matrices with coefficients in R by $M_{n \times n}(R)$ or $M_n(R)$. We denote by 1_n the multiplicative identity in $M_n(R)$. Also, for any subring R of $K \otimes_E E_v$, with v a place of E , let $\text{Her}_n(R)$ denote the space of Hermitian $n \times n$ -matrices with entries in R . Given $x \in \text{Her}_n(E)$,

$$x > 0$$

if $\sigma(x)$ is positive definite for every $\sigma \in \Sigma$.

2.1.1. *Adelic norms.* Let $|\cdot|_E$ denote the adelic norm on $E^\times \backslash \mathbb{A}_E^\times$ such that for all $a \in \mathbb{A}_E^\times$,

$$|a|_E = \prod_v |a|_v,$$

where the righthand product is over all places of E and where the absolute values are normalized so that

$$|v|_v = q_v^{-1},$$

$$q_v = \text{the cardinality of } \mathcal{O}_{E_v}/v\mathcal{O}_{E_v},$$

for all non-archimedean primes v of the totally real field E . Consequently for all $a \in E$,

$$\prod_{v \nmid \infty} |a|_v^{-1} = \prod_{v \in \Sigma} \sigma_v(a) \text{Sign}(\sigma_v(a)),$$

where the product is over all archimedean places v of the totally real field E . We denote by $|\cdot|_K$ the adelic norm on $K^\times \backslash \mathbb{A}_K^\times$ such that for all $a \in \mathbb{A}_K^\times$,

$$|a|_K = |a\bar{a}|_E.$$

For $a \in K$ and v a place of E , we let

$$|a|_v = |a\bar{a}|_v^{\frac{1}{2}}.$$

Given an element $a \in K$, we associate a with an element of $K \otimes \mathbb{R}$, via the embedding

$$a \mapsto (\sigma(a))_{\sigma \in \Sigma}.$$

For any field extension L/M , we write $\mathbf{N}_{L/M}$ to denote the norm from L to M . Given a \mathcal{O}_M -algebra R , the norm map $\mathbf{N}_{L/M}$ on L provides a group homomorphism

$$(\mathcal{O}_L \otimes R)^\times \rightarrow R^\times$$

in which $a \otimes r \mapsto \mathbf{N}_{L/M}(a)r$. When the fields are clear, we shall just write \mathbf{N} .

2.1.2. Exponential characters. For each archimedean place $v \in \Sigma$, denote by \mathbf{e}_v the character of E_v (i.e \mathbb{R}) defined by

$$\mathbf{e}_v(x_v) = e^{(2\pi i x_v)}$$

for all x_v in E_v . Denote by \mathbf{e}_∞ the character of $E \otimes \mathbb{R}$ defined by

$$\mathbf{e}_\infty((x_v)_{v \in \Sigma}) = \prod_{v \mid \infty} \mathbf{e}_v(x_v).$$

Following our convention from (1), we put

$$\mathbf{e}_\infty(a) = \mathbf{e}_\infty((\sigma(a))_{\sigma \in \Sigma}) = e^{2\pi i \operatorname{tr}_{E/\mathbb{Q}}(a)}$$

for all $a \in E$. For each finite place v of E dividing a prime q of \mathbb{Z} , denote by \mathbf{e}_v the character of E_v defined for each $x_v \in E_v$ by

$$\mathbf{e}_v(x_v) = e^{-2\pi i y}$$

where y is an element of \mathbb{Q} such that $\operatorname{tr}_{E_v/\mathbb{Q}_q}(x_v) - y \in \mathbb{Z}_p$. We denote by $\mathbf{e}_{\mathbb{A}_E}$ the character of \mathbb{A}_E defined by

$$\mathbf{e}_{\mathbb{A}_E}(x) = \prod_v \mathbf{e}_v(x_v)$$

for all $x = (x_v) \in \mathbb{A}_E$.

Remark 1. Note that for a $a \in E$, we identify a with the element $(\sigma_v(a))_v \in \mathbb{A}_E$, where $\sigma_v : E \hookrightarrow E_v$ is the embedding corresponding to v . Following this convention, we put

$$(3) \quad \mathbf{e}_{\mathbb{A}_E}(a) = \prod_v \mathbf{e}_v(\sigma_v(a)).$$

for all $a \in E$.

2.1.3. Spaces of functions. Given topological spaces X and Y , we let

$$\mathcal{C}(X, Y)$$

denote the space of continuous functions from X to Y .

2.2. Background concerning automorphic forms on unitary groups.

2.2.1. *Unitary groups of signature (n, n) .* We now recall basic information about unitary groups and automorphic forms on unitary groups. As we mentioned above, a more detailed discussion of unitary groups and automorphic forms on unitary groups appears in [Shi97, Shi00, Lan08, HLS06, EHLS]; the analogous background for the case of Hilbert modular forms is the main subject of [Kat78, Section 1].

The material in this section is similar to the material in [Eis11, Section 2.1]. Although we discussed embeddings of non-definite unitary groups of various signatures into unitary groups of signature (n, n) in [Eis11, Section 2.1], we shall be primarily concerned only with unitary groups of signature (n, n) and definite unitary groups in this paper; in the sequel, [Eis], we discuss pullbacks to non-definite unitary groups.

Let V be a vector space of dimension n over the CM field K , and let \langle, \rangle_V denote a positive definite hermitian pairing on V . Let $-V$ denote the vector space V with the negative definite hermitian pairing $-\langle, \rangle_V$. Let

$$W = 2V = V \oplus -V$$

$$\langle (v_1, v_2), (w_1, w_2) \rangle_W = \langle v_1, w_1 \rangle_V + \langle v_2, w_2 \rangle_{-V}.$$

The hermitian pairing \langle, \rangle_W defines an involution on $g \mapsto \tilde{g}$ on $\text{End}_K(W)$ by

$$\langle g(w), w' \rangle_W = \langle w, \tilde{g}(w') \rangle_W$$

(where w and w' denote elements of W). Note that this involution extends to an involution on $\text{End}_{K \otimes_E R}(V \otimes_E R)$ for any E -algebra R . We denote by U the algebraic group such that for any E -algebra R , the R -points of G are given by

$$U(R) = U(R, W) = \{g \in \text{GL}_{K \otimes_E R}(W \otimes_E R) | g\tilde{g} = 1\}.$$

Similarly, we define $U(R, V)$ to be the algebraic group associated to \langle, \rangle_V and $U(R, -V)$ to be the algebraic group associated to \langle, \rangle_{-V} . Note that $U(\mathbb{R})$ is of signature (n, n) . Also, note that the canonical embedding

$$V \oplus V \hookrightarrow W$$

induces an embedding

$$U(R, V) \times U(R, -V) \hookrightarrow U(R, W)$$

for all E -algebras R . When the E -algebra R over which we are working is clear from context or does not matter, we shall write $U(W)$ for $U(R, W)$, $U(V)$ for $U(R, V)$, and $U(-V)$ for $U(R, -V)$. We also sometimes write just U to denote $U(W)$.

We also have groups

$$GU(R) = GU(R, W) = \{g \in \text{GL}_{K \otimes_E R}(W \otimes_E R) | g\tilde{g} \in R^\times\}.$$

We use the notation ν to denote the similitude character

$$\begin{aligned}\nu : GU(R) &\rightarrow R^\times \\ g &\mapsto g\tilde{g}.\end{aligned}$$

When the E -algebra R over which we are working is clear from context or does not matter, we shall write $GU(W)$ for $GU(R, W)$. We shall also use the notation

$$G(R) = GU(R, W)$$

or write simply G or GU when the ring R is clear from context or does not matter. When $R = \mathbb{A}_E$ or $R = \mathbb{R}$, we write

$$G_+ := GU_+$$

to denote the subgroup of $G = GU$ consisting of elements such that the similitude factor at each archimedean place of $K \otimes_E R$ is positive.

For the space $W = V \oplus -V$ defined above, $U(W)$ and $GU(W)$ have signature (n, n) . So we will sometimes write $U(n, n)$ and $GU(n, n)$, respectively, to refer to these groups.

We write $W = V_d \oplus V^d$, where V_d and V^d denote the maximal isotropic subspaces

$$\begin{aligned}V^d &= \{(v, v) | v \in V\} \\ V_d &= \{(v, -v) | v \in V\}.\end{aligned}$$

Let P be the Siegel parabolic subgroup of $U(W)$ stabilizing V^d in $V_d \oplus V^d$ under the action of $U(W)$ on the right. Denote by M the Levi subgroup of P and by N the unipotent radical of P . Similarly, denote by GP the Siegel parabolic subgroup of $GU(W)$ stabilizing V^d in $V_d \oplus V^d$ under the action of $GU(W)$ on the right, and denote by GM the Levi subgroup of GP and by N the unipotent radical of GP . We also, similarly, denote by GP_+ the Siegel parabolic subgroup of GU_+ stabilizing V^d in $V_d \oplus V^d$ under the action of GU_+ on the right, and denote by GM_+ the Levi subgroup of GP_+ and by N the unipotent radical of GP_+ .

A choice of a basis e_1, \dots, e_n for V over K gives an identification of V with V^d (via $e_i \mapsto (e_i, e_i)$) and with V_d (via $e_i \mapsto (e_i, -e_i)$). The choice of a basis for V also identifies $\mathrm{GL}_K(V)$ with $\mathrm{GL}_n(K)$. With respect to the ordered basis $(e_1, e_1), \dots, (e_n, e_n), (e_1, -e_1), \dots, (e_n, -e_n)$ for W , M consists of the block diagonal matrices of the form

$$m(h) := \begin{pmatrix} {}^t\overline{h}^{-1} & \\ & h \end{pmatrix}$$

with $h \in \mathrm{GL}_n(K \otimes R)$, and GM consists of the block diagonal matrices of the form

$$m(h, \lambda) := \begin{pmatrix} {}^t\overline{h}^{-1} & \\ & \lambda h \end{pmatrix}$$

with $h \in \mathrm{GL}_n(K)$ and $\lambda \in E^\times$. Thus, the choice of basis e_1, \dots, e_n for V over K fixes identifications

$$\begin{aligned} M &\xrightarrow{\sim} \mathrm{GL}_K(V) \\ GM &\xrightarrow{\sim} \mathrm{GL}_K(V) \times E^\times. \end{aligned}$$

Note that these isomorphisms extend to isomorphisms

$$(4) \quad M(R) \xrightarrow{\sim} \mathrm{GL}_{K \otimes_E R}(V \otimes_E R)$$

$$(5) \quad GM(R) \xrightarrow{\sim} \mathrm{GL}_{K \otimes_E R}(V \otimes_E R) \times R^\times$$

for each E -algebra R .

We fix a Shimura datum $(G, X(W))$ and a corresponding Shimura variety $\mathrm{Sh}(W) = \mathrm{Sh}(U(n, n))$, according to the conditions in [HLS06, Eis12]. Note that the symmetric domain $X(W)$ is holomorphically isomorphic to the tube domain consisting of $[E : \mathbb{Q}]$ copies of

$$\mathcal{H}_n = \{z \in M_{n \times n}(\mathbb{C}) \mid i({}^t \bar{z} - z) > 0\}.$$

When we need to emphasize over which ring R we work, we sometimes write $\mathrm{Sh}(R)$. Let \mathcal{K}_∞ be the stabilizer in $G(\mathbb{R})$ of the point $i \cdot 1_n$. So $\prod_{\sigma \in \Sigma} \mathcal{K}_\infty$ is the stabilizer in $\prod_{\sigma \in \Sigma} G(\mathbb{R})$ of the point

$$(6) \quad \mathbf{i} = (i \cdot 1_n)_{\sigma \in \Sigma} \in \prod_{\sigma \in \Sigma} \mathcal{H}_n.$$

Note that we can identify $G(\mathbb{R})/\mathcal{K}_\infty$ with \mathcal{H}_n . Given a compact open subgroup \mathcal{K} of $G(\mathbb{A}_f)$, denote by ${}_{\mathcal{K}}\mathrm{Sh}(W)$ the Shimura variety whose complex points are given by

$$G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / \mathcal{K}.$$

This Shimura variety is a moduli space for abelian varieties together with a polarization, an endomorphism, and a level structure (dependent upon the choice of \mathcal{K}). Note that ${}_{\mathcal{K}}\mathrm{Sh}(W)$ consists of copies of quotients of \mathcal{H}_n .

When we are working with some other group H , we write $\mathrm{Sh}(H)$ instead of $\mathrm{Sh}(W)$.

2.2.2. Automorphic Forms on unitary groups. Automorphic forms on unitary groups are typically discussed from any of the following three perspectives (which are equivalent over \mathbb{C}):

- (1) Functions on a unitary group that satisfy an automorphy condition
- (2) C^∞ - (or holomorphic) functions on a hermitian symmetric space (analogue of the upper half plane) that satisfy an automorphy condition
- (3) Sections of a certain vector bundle over a moduli space (a Shimura variety) parametrizing abelian varieties together with a polarization, endomorphism, and level structure

Which perspective is most natural depends upon context. In this paper, we shall need all three perspectives. In [Eis12], we provided a detailed discussion of automorphic forms and the relationships between different approaches to defining them. In this section, we summarize only the basic facts needed for this paper.

The relationship between the first two approaches to automorphic forms is reviewed in [Eis11, p. 9], as well as other references, such as [Shi00, A8] and [Shi97]. The relationship between the second two approaches to automorphic forms is discussed in [Eis12] and is similar to the analogous relationship for modular forms given in [Kat73].

Note that an automorphic form f on $U(n, n)$ has a weight, which is a representation ρ of $\mathrm{GL}_n \times \mathrm{GL}_n$. In the special case where this representation is of the form

$$\rho(a, b) = \det(a)^{k+\nu} \det(b)^{-\nu},$$

we shall say f is an automorphic form of weight (k, ν) .

As explained in [Lan08, Lan12], for the unitary groups of signature (n, n) , there is a higher-dimensional analogue of the Tate curve (which we call the “Mumford object” in [Eis12, Eis11]), and so in analogue with the case for modular forms evaluated at the Tate curve, one obtains an algebraic q -expansion by evaluating an automorphic form at the Mumford object. Like in the case of modular forms, the coefficients of the algebraically defined q -expansion of an automorphic form f over \mathbb{C} agree with the (analytically defined) Fourier coefficients of f [Lan12]. Also, like in the case of modular forms, there is a q -expansion principle for automorphic forms on unitary groups [Lan08, Prop 7.1.2.15]; note that the q -expansion principle for automorphic forms over a Shimura variety requires the evaluation of an automorphic form at one cusp of each connected component. As explained in [Hid04, Section 8.4], to apply the q -expansion principle, it is enough to check the cusps parametrized by points of $GM_+(\mathbb{A}_E)$. (The author is grateful to thank Kai-Wen Lan for explaining this to her.) We shall say “a cusp $m \in GM_+(\mathbb{A}_E)$ ” to mean “the cusp corresponding to the point m .” Note that the q -expansion of an automorphic form at a cusp $m(h, \lambda)$ is a sum of the form

$$\sum_{\beta \in L_{m(h, \lambda)}} a(\beta) q^\beta,$$

where $L_{m(h, \lambda)}$ is a lattice in $\mathrm{Her}_n(E)$ dependent upon the choice of the cusp $m(h, \lambda)$. We sometimes also write

$$\sum_{\beta \in \mathrm{Her}_n(E)} a(\beta) q^\beta,$$

when we do not need to make the cusp explicit; in this case, we know that the coefficients $a(\beta)$ are zero outside of some lattice in $\mathrm{Her}_n(E)$ (namely, the lattice corresponding to the unspecified cusp).

Note that throughout the paper, all cusps m and corresponding lattices $L_m \subseteq \text{Her}_n(K)$ determined by m are chosen so that the elements of L_m have p -integral coefficients.¹

3. EISENSTEIN SERIES ON UNITARY GROUPS

In this section, we introduce certain Eisenstein series on unitary groups of signature (n, n) . These Eisenstein series are related to the ones discussed in [Eis11, Shi97, Kat78].

For $k \in \mathbb{Z}$ and $\nu = (\nu(\sigma))_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$, we denote by $\mathbf{N}_{k,\nu}$ the function

$$\begin{aligned} \mathbf{N}_{k,\nu} : K^\times &\rightarrow K^\times \\ b &\mapsto \prod_{\sigma \in \Sigma} \sigma(b)^{k+2\nu(\sigma)} (\sigma(b)\bar{\sigma}(b))^{-\nu(\sigma)}. \end{aligned}$$

Note that for all $b \in \mathcal{O}_E^\times$,

$$\mathbf{N}_{k,\nu}(b) = \mathbf{N}_{E/\mathbb{Q}}^k(b).$$

Theorem 2. *Let R be an \mathcal{O}_K -algebra, let $\nu = (\nu(\sigma)) \in \mathbb{Z}^\Sigma$, and let $k \geq n$ be an integer. Let*

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

be a locally constant function supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ that satisfies

$$(7) \quad F(ex, \mathbf{N}_{K/E}(e^{-1})y) = \mathbf{N}_{k,\nu}(e)F(x, y)$$

for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$. There is an automorphic form $G_{k,\nu,F}$ (on $U(n, n)$) of weight (k, ν) defined over R whose q -expansion at a cusp $m \in GM_+(\mathbb{A}_E)$ is of the form $\sum_{0 < \beta \in L_m} c(\beta)q^\beta$ (where L_m is the lattice in $\text{Her}_n(K)$ determined by m), with $c(\beta)$ a finite \mathbb{Z} -linear combination of terms of the form

$$F(a, \mathbf{N}_{K/E}(a)^{-1}\beta) \mathbf{N}_{k,\nu}(a^{-1} \det \beta) \mathbf{N}_{E/\mathbb{Q}}(\det \beta)^{-n}$$

(where the linear combination is a sum over a finite set of p -adic units $a \in K$ dependent upon β and the choice of cusp $m \in GM$). When $R = \mathbb{C}$, these are the Fourier coefficients at $s = \frac{k}{2}$ of the C^∞ -automorphic form $G_{k,\nu,F}(z, s)$ (which is holomorphic at $s = \frac{k}{2}$) that will be defined in Lemma 9.

(Above, the elements of $(\mathcal{O}_E \otimes \mathbb{Z}_p)^\times$ in $M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ are viewed as homomorphisms, i.e. multiplication by an element of $(\mathcal{O}_E \otimes \mathbb{Z}_p)^\times$, so as diagonal matrices in $M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$. Also, note that when $\det \beta = 0$, the coefficient of q^β is 0, so we can restrict the discussion to F with support in $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times \text{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$.)

¹Even without this choice for m and L_m , which we did not make *a priori* in [Eis11], we could force the Fourier coefficients at all the non- p -integral elements of $\text{Her}_n(K)$ to be zero, simply by our choice of a Siegel section at p later in this paper. In fact, in [Eis11], our choice of Siegel sections at p forced the Fourier coefficients at all the non- p -integral elements of $\text{Her}_n(K)$ to be zero.

Proof. By an argument similar to Katz's argument at the beginning of the proof of [Kat78, Theorem (3.2.3)], every locally constant R -valued function F supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ that satisfies Equation (7) is an R -linear combination of \mathcal{O}_K -valued functions F supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ that satisfy Equation (7). So it is enough to prove the theorem for \mathcal{O}_K -valued functions F .

Now, if we can construct an automorphic form satisfying the conditions of the theorem over $R = \mathbb{C}$, then by the q -expansion principle [Lan08, Prop 7.1.2.15], the case over R will follow for any \mathcal{O}_K -subalgebra R (in particular, for $R = \mathcal{O}_K$) of \mathbb{C} . By [Lan12], it is sufficient to show that there is a \mathbb{C} -valued C^∞ -automorphic form $G_{k,\nu,F}$ of weight (k, ν) holomorphic at $s = \frac{k}{2}$, whose Fourier coefficients (at $s = \frac{k}{2}$) are as in the statement of the theorem. We will spend the remainder of this section (i.e. all of Section 3.1) constructing such an automorphic form. \blacksquare

3.1. Construction of a C^∞ -automorphic form over \mathbb{C} whose Fourier coefficients meet the conditions of Theorem 2. In this section, we construct the C^∞ -automorphic form $G_{k,\nu,F}$ necessary to complete the proof of Theorem 2.

Let \mathfrak{m} be an ideal that divides p^∞ . Let χ be a unitary Hecke character of type A_0

$$\chi : \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times$$

of conductor \mathfrak{m} , i.e.

$$\chi_v(a) = 1$$

for all finite primes v in K and all $a \in K_v^\times$ such that

$$a \in 1 + \mathfrak{m}_v \mathcal{O}_{K_v}.$$

Let $\nu(\sigma)$ and $k(\sigma)$, $\sigma \in \Sigma$, denote integers such that the infinity type of χ is

$$(8) \quad \prod_{\sigma \in \Sigma} \sigma^{-k(\sigma) - 2\nu(\sigma)} (\sigma \cdot \bar{\sigma})^{\frac{k(\sigma)}{2} + \nu(\sigma)}.$$

For any $s \in \mathbb{C}$, we view $\chi \cdot |\cdot|_K^{-s} \otimes |\cdot|_E^{-ns}$ as a character of the parabolic subgroup $GP_+(\mathbb{A}_E) = GM_+(\mathbb{A}_E)N(\mathbb{A}_E) \subseteq G_+(\mathbb{A}_E)$ via the composition of maps

$$\begin{aligned} GP(\mathbb{A}_E) &\xrightarrow{\text{mod } N(\mathbb{A}_E)} GM(\mathbb{A}_E) \xrightarrow{\text{map in (5)}} \text{GL}_{\mathbb{A}_K}(V \otimes_E \mathbb{A}_E) \times \text{GL}_1(\mathbb{A}_E) \\ &\xrightarrow{(h, \lambda) \mapsto |\lambda|_E^{-ns} \cdot \chi(\det h) |\det h|_K^{-s}} \mathbb{C}^\times. \end{aligned}$$

Consider the induced representation

$$\begin{aligned} I(\chi, s) &= \text{Ind}_{GP_+(\mathbb{A}_E)}^{G_+(\mathbb{A}_E)} (\chi \cdot |\cdot|_K^{-s} \otimes |\nu(\cdot)|_K^{-ns/2}) \\ (9) \quad &\cong \otimes_v \text{Ind}_{GP_+(E_v)}^{G_+(E_v)} (\chi_v |\cdot|_v^{-2s} \otimes |\nu(\cdot)|_v^{-ns}), \end{aligned}$$

where the product is over all places of E .

Given a section $f \in I(\chi, s)$, the Siegel Eisenstein series associated to f is the \mathbb{C} -valued function of G defined by

$$E_f(g) = \sum_{\gamma \in GP_+(E) \backslash G_+(E)} f(\gamma g)$$

This function converges for $\Re(s) > 0$ and can be continued meromorphically to the entire complex plane.

Remark 3. As in [Eis11], if we were working with normalized induction, then the function would converge for $\Re(s) > \frac{n}{2}$, but we have absorbed the exponent $\frac{n}{2}$ into the exponent s . (Our choice not to include the modulus character at this point is equivalent to shifting the plane on which the function converges by $\frac{n}{2}$.)

All the poles of E_f are simple and there are at most finitely many of them. Details about the poles are given in [Tan99].

As we noted in [Eis11, Section 2.2.4], if the Siegel section f factors as $f = \otimes_v f_v$, then E_f has a Fourier expansion such that for all $h \in \mathrm{GL}_n(K)$ and $m \in \mathrm{Her}_n(K)$,

$$E_f \left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t \bar{h}^{-1} & 0 \\ 0 & h \end{pmatrix} \right) = \sum_{\beta \in \mathrm{Her}_n(K)} c(\beta, h; f) \mathbf{e}_{\mathbb{A}_E}(\mathrm{tr}(\beta m)),$$

with $c(\beta, h; f)$ a complex number dependent only on the choice of section f , the hermitian matrix $\beta \in \mathrm{Her}_n(K)$, h_v for finite places v , and $(h \cdot {}^t \bar{h})_v$ for archimedean places v of E .

By [Shi97, Sections 18.9, 18.10], the Fourier coefficients of the Siegel sections $f = \otimes_v f_v$ that we will choose below are products of local Fourier coefficients determined by the local sections f_v . More precisely, for each $\beta \in \mathrm{Her}_n(K)$,

$$c(\beta, h; f) = C(n, K) \prod_v c_v(\beta, h; f),$$

where

$$(10) \quad c_v(\beta, h; f) =$$

$$\int_{\mathrm{Her}_n(K \otimes E_v)} f_v \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & m_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t \bar{h}_v^{-1} & 0 \\ 0 & h_v \end{pmatrix} \right) \mathbf{e}_v(-\mathrm{tr}(\beta_v m_v)) dm_v,$$

$$(11) \quad C(n, K) = 2^{n(n-1)[E:\mathbb{Q}]/2} |D_E|^{-n/2} |D_K|^{-n(n-1)/4},$$

D_E and D_K are the discriminants of K and E respectively, $\beta_v = \sigma_v(\beta)$ for each place v of E , and d_v denotes the Haar measure on $\mathrm{Her}_n(K_v)$ such that:

$$(12) \quad \int_{\mathrm{Her}_n(\mathcal{O}_{K \otimes E_v})} d_v x = 1, \text{ for each finite place } v \text{ of } E$$

$$d_v x := \left| \bigwedge_{j=1}^n dx_{jj} \bigwedge_{j < k} (2^{-1} dx_{jk} \wedge d\bar{x}_{jk}) \right|, \text{ for each archimedean place } v \text{ of } E.$$

(In Equation (12), x denotes the matrix whose ij -th entry is x_{ij} .)

Below, we recall [Eis11, Lemma 19], which explains how the Fourier coefficients $c(\beta, h; f)$ transform when we change the point h .

Lemma 4 (Lemma 19 in [Eis11]). *For each $h \in \mathrm{GL}_n(\mathbb{A}_K)$, $\lambda \in \mathbb{A}_E^\times$, and $\beta \in \mathrm{Her}_n(K)$,*

$$(13) \quad c\left(\beta, \begin{pmatrix} {}^t\bar{h}^{-1} & 0 \\ 0 & \lambda h \end{pmatrix}; f\right) = \chi(\det(\overline{\lambda h})^{-1}) \left| \det\left(\overline{(\lambda h)}^{-1} \cdot (\lambda h)^{-1}\right) \right|_E^{n-s} |\lambda|_E^{-ns} c(\lambda^{-1t}\bar{h}^{-1}\beta h^{-1}, 1_n; f).$$

Proof. Let $\eta = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. Let $m(h, \lambda)$ denote the matrix $\begin{pmatrix} {}^t\bar{h}^{-1} & 0 \\ 0 & \lambda h \end{pmatrix}$. Observe that for any $n \times n$ matrix m ,

$$\begin{aligned} \eta \cdot m(h, \lambda) \cdot \eta^{-1} &= m(\lambda^{-1t}\bar{h}^{-1}, \lambda) \\ m(h, \lambda)^{-1} \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot m(h, \lambda) &= \begin{pmatrix} 1 & \lambda^t\bar{h}mh \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \eta \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot m(h, \lambda) &= (\eta \cdot m(h, \lambda) \cdot \eta^{-1}) \eta \left(m(h, \lambda)^{-1} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} m(h, \lambda) \right) \\ &= m(\lambda^{-1t}\bar{h}^{-1}, \lambda) \eta \begin{pmatrix} 1 & \lambda^t\bar{h}mh \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

So for any place v of E and section $f_v \in \mathrm{Ind}_{GP(E_v)}^{G_+(E_v)}(\chi, s)$,

$$(14) \quad f_v \left(\eta \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} m(h_v, \lambda) \right)$$

$$(15) \quad = \chi_v \left(\det \overline{\lambda_v h_v}^{-1} \right) \left| \det \overline{\lambda_v h_v}^{-1} \right|_v^{-2s} |\lambda|_v^{-ns} f_v \left(\eta \begin{pmatrix} 1 & \lambda^t\bar{h}_v m h_v \\ 0 & 1 \end{pmatrix} \right).$$

The lemma now follows from Equation (14) and the fact that the Haar measure d_v satisfies $d_v(\lambda h_v x^t \bar{h}_v) = |\det(\lambda_v {}^t\bar{h}_v \cdot h_v)|_v^n d_v(x)$ for each place v of E . \blacksquare

So

$$\begin{aligned} c\left(\beta, \begin{pmatrix} \lambda^{-1t}\bar{h}^{-1} & 0 \\ 0 & h \end{pmatrix}; f\right) &= \chi(\lambda^n) |\lambda^{2n}|_E^{n-s} |\lambda|_E^{2ns} c\left(\beta, \begin{pmatrix} {}^t\bar{h}^{-1} & 0 \\ 0 & \lambda h \end{pmatrix}; f\right) \\ &= |\lambda^{2n^2}|_E \chi(\lambda^n) c\left(\beta, \begin{pmatrix} {}^t\bar{h}^{-1} & 0 \\ 0 & \lambda h \end{pmatrix}; f\right). \end{aligned}$$

Below, we choose more specific Siegel sections $f = \otimes_v f_v$ and compute the corresponding Fourier coefficients.

3.1.1. *The Siegel section at ∞ .* In this section, we define a section $f_\infty^{k,\nu} = f_\infty^{k,\nu}(\bullet; i \cdot 1_n, \chi, s) \in \otimes_{v|\infty} \text{Ind}_{GP_+(E_v)}^{G_+(E_v)} (\chi_v ||_v^{-2s} \otimes |\nu(\cdot)|_E^{-ns})$.

For each $\alpha = \prod_{v|\infty} \alpha_v \in \prod_{v|\infty} G(E_v)$, we write α_v in the form $\begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$ with a_v, b_v, c_v , and d_v $n \times n$ matrices. Each element $\alpha \in G(E_v)$ acts on $z = \prod_{v|\infty} z_v \in \prod_{v|\infty} \mathcal{H}_n$ by

$$\begin{aligned} \alpha_v(z_v) &= (a_v z + b_v)(c_v z + d_v)^{-1} \\ \alpha(z) &= \prod_{v|\infty} \alpha_v(z_v). \end{aligned}$$

Let

$$\begin{aligned} \lambda_{\alpha_v}(z_v) &= \lambda(\alpha_v, z_v) = \overline{c_v} \cdot {}^t z_v + \overline{d_v} \\ \lambda_\alpha(z) &= \lambda(\alpha, z) = \prod_{v|\infty} \lambda_{\alpha_v}(z_v) \\ \mu_{\alpha_v}(z_v) &= \mu(\alpha_v, z_v) = c_v \cdot z_v + d_v \\ \mu_\alpha(z) &= \mu(\alpha, z) = \prod_{v|\infty} \mu_{\alpha_v}(z_v). \end{aligned}$$

(These are the canonical automorphy factors. Properties of them are discussed in, for example, [Shi00, Section 3.3].) We have chosen our notation to be similar to the notation used in [Shi97, Shi00]. We write

$$\begin{aligned} j_{\alpha_v}(z_v) &= j(\alpha_v, z_v) = \det \mu_{\alpha_v}(z_v) \\ j_\alpha(z) &= j(\alpha, z) = \prod_{v|\infty} j_{\alpha_v}(z_v). \end{aligned}$$

Note that

$$\begin{aligned} (16) \quad \det(\lambda_{\alpha_v}(z_v)) &= \det(\overline{\alpha_v}) \nu(\alpha_v)^{-n} j_{\alpha_v}(z_v) \\ (17) \quad &= \det(\alpha_v)^{-1} \nu(\alpha_v)^n j_{\alpha_v}(z_v). \end{aligned}$$

So

$$|\det(\lambda_{\alpha_v}(z_v))| = |j_{\alpha_v}(z_v)|.$$

Consistent with the notation in [Shi97], we shall write

$$j_\alpha^{k,\nu}(z) = j_\alpha(z)^{k+\nu} \det(\lambda_\alpha(z))^{-\nu}.$$

By Equations (16) and (17), we see that

$$\begin{aligned} j_\alpha^{k,\nu}(z) &= (\det(\overline{\alpha}) \nu(\alpha)^{-n})^{-\nu} j_\alpha(z)^k \\ &= (\det(\alpha)^{-1} \nu(\alpha)^n)^{-\nu} j_\alpha(z)^k \end{aligned}$$

Note that if $\beta = \prod_{v|\infty} \beta_v$ is also an element of $\prod_{v|\infty} G(E_v)$, then

$$(18) \quad \lambda(\beta_v \alpha_v, z_v) = \lambda(\beta_v, \alpha_v z_v) \lambda(\alpha_v, z_v)$$

$$(19) \quad \mu(\beta_v \alpha_v, z_v) = \mu(\beta_v, \alpha_v z_v) \mu(\alpha_v, z_v).$$

Consistent with the notation in [Shi00, Section 3], we define functions η and δ on \mathcal{H}_n by

$$\begin{aligned}\eta(z) &= i \left({}^t \bar{z} - z \right) \\ \delta(z) &= \det \left(\frac{1}{2} \eta(z) \right)\end{aligned}$$

for each $z \in \mathcal{H}_n$. So

$$\begin{aligned}\eta(i \cdot 1_n) &= 2 \cdot 1_n \\ \delta(i \cdot 1_n) &= 1.\end{aligned}$$

We also write η and δ to denote the functions $\prod_{\sigma \in \Sigma} \eta$ and $\prod_{\sigma \in \Sigma} \delta$, respectively, on $\prod_{\sigma \in \Sigma} \mathcal{H}_n$. So $\delta(\mathbf{i}) = 1$. Also, note that

$$\begin{aligned}\delta(\alpha z) &= \nu(\alpha)^n |j_\alpha(z)|^{-2} \delta(z) \\ &= \nu(\alpha)^n |j_\alpha(z) \det(\lambda_\alpha(z))|^{-1} \delta(z).\end{aligned}$$

Continuing to use the notation in [Shi00, Sections 3 and 5], given $(k, \nu) = \prod_{v| \infty} (k_v, \nu_v) \in (\mathbb{Z} \times \mathbb{Z})^\Sigma$, we define functions $f|_{k, \nu}$ and $f|_{k, \nu}$ on $\prod_{\sigma \in \Sigma} \mathcal{H}_n$ by

$$\begin{aligned}(f|_{k, \nu} \alpha)(z) &= j_\alpha^{k, \nu}(z)^{-1} f(\alpha z) \\ f|_{k, \nu} \alpha &= f|_{k, \nu} \left(\nu(\alpha)^{-\frac{1}{2}} \alpha \right)\end{aligned}$$

for each \mathbb{C} -valued function f on \mathcal{H}_n , point $z \in \mathcal{H}_n$, and element $\alpha \in G$. Note that $\nu(\alpha)^{-\frac{1}{2}} \alpha \in U(\eta_n)$, and if $\nu(\alpha_v) = 1$ for all $v \in \Sigma$, then

$$f|_{k, \nu} \alpha = f|_{k, \nu}.$$

More generally, for each function f on $\prod_{\sigma \in \Sigma} \mathcal{H}_n$ with values in some representation (V, ρ) of $\prod_{\sigma \in \Sigma} \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, we define functions $f|_\rho$ and $f|_\rho$ on \mathcal{H}_n by

$$\begin{aligned}(f|_\rho \alpha)(z) &= \rho(\mu_\alpha(z), \lambda_\alpha(z))^{-1} f(\alpha z) \\ f|_\rho \alpha &= f|_\rho \left(\nu(\alpha)^{-\frac{1}{2}} \alpha \right).\end{aligned}$$

Note that we also use the notation $f|$ and $f|$ when we are working with just one copy of \mathcal{H}_n , rather than $[E : \mathbb{Q}]$ copies of \mathcal{H}_n at once.

We define

$$f_\infty^{k, \nu} = \otimes f_v^{k, \nu} \in (\bullet; i \cdot 1_n, \chi, s) \in \otimes_{v| \infty} \mathrm{Ind}_{GP_+(E_v)}^{G_+(E_v)} (\chi_v |_v^{-2s} \otimes |\nu(\cdot)|_E^{-ns})$$

by

$$\begin{aligned}f_\infty^{k, \nu}(\alpha; i \cdot 1_n, \chi, s) &= \left(\delta^{s - \frac{k}{2}} |_{k, \nu} \alpha \right) (i \cdot 1_n) \\ &= j_{\nu(\alpha)^{-1/2} \alpha}^{k, \nu} (i \cdot 1_n)^{-1} \left| j_{\nu(\alpha)^{-1/2} \alpha} (i \cdot 1_n)^{-2} \nu \left(\nu(\alpha)^{-1/2} \alpha \right)^n \right|^{s - \frac{k(\sigma_v)}{2}} \\ &= j_{\nu(\alpha)^{-1/2} \alpha}^{k, \nu} (i \cdot 1_n)^{-1} \left| j_{\nu(\alpha)^{-1/2} \alpha} (i \cdot 1_n)^{-2} \right|^{s - \frac{k}{2}}.\end{aligned}$$

Given $\alpha \in G$, we also define a function $f_\infty^{k,\nu}(\alpha; \bullet, \chi, s)$ on \mathcal{H}_n by

$$\begin{aligned} f_\infty^{k,\nu}(\alpha; z, \chi, s) &= \left(\delta^{s-\frac{k}{2}}|_{k,\nu}\alpha \right)(z) \\ &= j_{\nu(\alpha)^{-1/2}\alpha}^{k,\nu}(z)^{-1} \left| j_{\nu(\alpha)^{-1/2}\alpha}(z)^{-2} \right|^{s-\frac{k}{2}} \delta(z)^{s-\frac{k}{2}}. \end{aligned}$$

By Equations (18) and (19), we see that if $g \in G$ is such that

$$g(\mathbf{i}) = z,$$

then for each $\alpha \in G$,

$$f_\infty^{k,\nu}(\alpha g; i \cdot 1_n, \chi, s) = f_\infty^{k,\nu}(\alpha; z, \chi, s) f_\infty^{k,\nu}(g; i \cdot 1_n, \chi, s) \delta(z)^{\frac{k}{2}-s}.$$

For $k \in \mathbb{Z}$ and $\nu = (\nu_v)_{v \in \Sigma} \in \mathbb{Z}^\Sigma$, $f_\infty^{k,\nu}(\alpha; \bullet, \chi, s)$ is a holomorphic function on \mathcal{H}_n at $s = \frac{k}{2}$.

3.1.2. The Fourier coefficients at archimedean places of E . When there is an integer k such that

$$s = \frac{k}{2} = \frac{k(\sigma)}{2} \text{ for all } \sigma \in \Sigma$$

(i.e. when $f_\infty^{k,\nu}(\alpha; z, \chi, s)$ is a holomorphic function of $z \in \mathcal{H}_n$), [Shi83, Equation (7.12)] describes the archimedean Fourier coefficients precisely:

$$\begin{aligned} &c_v \left(\beta, 1_n; f_v^{k,\nu} \left(\bullet; i 1_n, \chi, \frac{k}{2} \right) \right) \\ (20) \quad &= 2^{(1-n)n_i - nk} (2\pi)^{nk} \left(\pi^{n(n-1)/2} \prod_{t=0}^{n-1} \Gamma(k-t) \right)^{-1} \sigma_v(\det \beta)^{k-n} \mathbf{e}(\text{itr}(\sigma_v(\beta))), \end{aligned}$$

for each archimedean place v of E . Observe that when $k \geq n$,

$$\prod_{v|\infty} c_v \left(\beta, h; f_v^{k,\nu} \left(\bullet; i 1_n, \chi, \frac{k}{2} \right) \right) = 0,$$

unless $\det(\beta) \neq 0$ and $\det(h) \neq 0$, i.e. unless β is of rank n . Also, note that in our situation, β will be in $\text{Her}_n(K)$, so $\prod_{v \in \Sigma} \mathbf{e}(\text{itr}(\sigma_v(\beta))) = \mathbf{e}(ib)$ for some $b \in \mathbb{Q}$, so $\prod_{v \in \Sigma} \mathbf{e}(\text{itr}(\sigma_v(\beta))) = \mathbf{e}(ib)$ is a root of unity.

3.1.3. Siegel Sections at p . We work with Siegel sections at p that are similar to the ones in [Eis11]. (To account for a similitude factor, we multiply the Siegel section from [Eis11] by $|\nu(g)|_p^{-ns}$.)

Lemma 5 (Lemma 10, [Eis11]). *Let Γ be a compact open subset of $\prod_{v \in \Sigma} \text{GL}_n(\mathcal{O}_{E_v})$, and let \tilde{F} be a locally constant Schwartz function*

$$\begin{aligned} \tilde{F} : \prod_{v \in \Sigma} \left(\text{Hom}_{K_v}(V_v, V_{d,v}) \oplus \text{Hom}_{K_v}(V_v, V_v^d) \right) &\rightarrow R \\ (X_1, X_2) &\mapsto \tilde{F}(X_1, X_2) \end{aligned}$$

(with R a subring of \mathbb{C}) whose support in the first variable is Γ and such that

$$(21) \quad \tilde{F}(X, {}^t X^{-1} Y) = \prod_{v \in \Sigma} \chi_v(\det(X)) \tilde{F}(1, Y)$$

for all X in Γ and Y in $\prod_{v \in \Sigma} M_{n \times n}(E_v)$.² There is a Siegel section $f^{P\tilde{F}}(-X, Y)$ at p whose Fourier coefficient at $\beta \in M_{n \times n}(E_v)$ is

$$c(\beta, 1; f^{P\tilde{F}}(-X, Y)) = \text{volume}(\Gamma) \cdot \tilde{F}(1, {}^t \beta).$$

Note that PF stands for “partial Fourier transform.” (We use that notation to be consistent with [Eis11], but we do not need to discuss partial Fourier transforms here.)

As a direct consequence of Lemma 5, we obtain the following corollary:

Corollary 6. *For any locally constant Schwartz function \tilde{F} satisfying the conditions of Lemma 5 for some Γ with positive volume, there is a Siegel section $f_{\tilde{F}}$ in $\otimes_{v \in \Sigma} \text{Ind}_{P(E_v)}^{G(E_v)}(\chi_v \cdot |\cdot|^{-2s})$ whose local (at p) Fourier coefficient at β is $\tilde{F}(1, {}^t \beta)$.*

Furthermore, as we explain in Corollary 7, we can significantly weaken the conditions placed on F in Corollary 6.

Corollary 7. *Let k be a positive integer. Let \tilde{F} be a locally constant Schwartz function*

$$\tilde{F} : \left(\prod_{v \in \Sigma} (M_{n \times n}(\mathcal{O}_{E_v}) \times M_{n \times n}(\mathcal{O}_{E_v})) \right) \rightarrow R$$

whose support lies in $\prod_{v \in \Sigma} (\text{GL}_n(\mathcal{O}_{E_v}) \times M_{n \times n}(\mathcal{O}_{E_v}))$ and which satisfies

$$\tilde{F}(e, {}^t e^{-1} y) = \mathbf{N}_{E/\mathbb{Q}}(\det e)^k \tilde{F}(1, y),$$

for all $e \in \text{GL}_n(\mathcal{O}_E)$ contained in the support Γ in the first variable of \tilde{F} . Suppose, furthermore, that Γ has positive volume. Then there is a Siegel section $f_{\tilde{F}} \in \otimes_{v \in \Sigma} \text{Ind}_{P(E_v)}^{G(E_v)}(\chi_v \cdot |\cdot|^{-2s})$ whose local (at p) Fourier coefficient at β is $\tilde{F}(1, {}^t \beta)$.

Proof. Let \tilde{F} be a locally constant Schwartz function

$$\tilde{F} : \prod_{v \in \Sigma} (M_{n \times n}(\mathcal{O}_{E_v}) \times M_{n \times n}(\mathcal{O}_{E_v})) \rightarrow R$$

whose support lies in $\prod_{v \in \Sigma} (\text{GL}_n(\mathcal{O}_{E_v}) \times M_{n \times n}(\mathcal{O}_{E_v}))$ and which satisfies

$$(22) \quad \tilde{F}(e, {}^t e^{-1} y) = \mathbf{N}_{E/\mathbb{Q}}(\det e)^k \tilde{F}(1, y),$$

²The version of the righthand side of Equation (21) appearing in [Eis11, Lemma 10] reads “ $\chi_1 \chi_2^{-1}(\det(X)) F(1, Y)$.” The characters denoted χ_1 and χ_2 in [Eis11] have the property that $\chi_1 \chi_2^{-1}(a) = \prod_{v \in \Sigma} \chi_v(a)$ for all $a \in \prod_{v \in \Sigma} \mathcal{O}_{E_v}$. The function denoted by \tilde{F} in the current paper is denoted by F in [Eis11].

for all $e \in \mathrm{GL}_n(\mathcal{O}_E)$ contained in the support in the first variable of \tilde{F} . Then since \tilde{F} is locally constant, has compact support, and satisfies Equation (22), there is a unitary Hecke character χ whose infinity type is as in Expression (8) and such that the conductor $\mathfrak{m} = p^d$ for d a sufficiently large positive integer) so that

$$\tilde{F} = a_1 F_1 + \cdots + a_l F_l$$

for some positive integer l , $a_1, \dots, a_l \in R$, and functions F_1, \dots, F_l meeting the conditions of Corollary 6 (all for this *same character* χ but possibly with *different supports* $\Gamma_1, \dots, \Gamma_l$, respectively, in the first variable).

Now, we define

$$f_{\tilde{F}} := a_1 f_{F_1} + \cdots + a_l f_{F_l},$$

where f_{F_1}, \dots, f_{F_l} are the Siegel sections obtained in Corollary 6. Then $f_{\tilde{F}}$ is a linear combination of elements of the module $\otimes_{v \in \Sigma} \mathrm{Ind}_{P(E_v)}^{G(E_v)} (\chi_v \cdot |\cdot|^{-2s})$. So $f_{\tilde{F}}$ is itself an element of $\otimes_{v \in \Sigma} \mathrm{Ind}_{P(E_v)}^{G(E_v)} (\chi_v \cdot |\cdot|^{-2s})$. Now, the Fourier coefficient of a sum of Siegel sections is the sum of the Fourier coefficients of these Siegel sections. So the Fourier coefficient at β of $f_{\tilde{F}}$ is

$$a_1 F_1(1, {}^t\beta) + \cdots + a_l F_l(1, {}^t\beta) = \tilde{F}(1, {}^t\beta).$$

■

3.1.4. Siegel Sections away from p and ∞ . We use the same Siegel sections at places $v \nmid p\infty$ as in [Eis11]. We now recall the key properties of these Siegel sections, which are described in more detail in [Shi97, Section 18].

Let \mathfrak{b} be an ideal in \mathcal{O}_E prime to p . For each finite place v prime to p , there is a Siegel section $f_v^{\mathfrak{b}} = f_v^{\mathfrak{b}}(\bullet; \chi_v, s) \in \mathrm{Ind}_{P(E_v)}^{G(E_v)} (\chi_v, s)$ with the following property: By [Shi97, Proposition 19.2], whenever the Fourier coefficient $c(\beta, m(1); f_v^{\mathfrak{b}})$ is non-zero,

$$(23) \quad \prod_{v \nmid p\infty} c(\beta, m(1); f_v^{\mathfrak{b}}) = N_{E/\mathbb{Q}}(\mathfrak{b}\mathcal{O}_E)^{-n^2} \prod_{i=0}^{n-1} L^p(2s - i, \chi_E^{-1} \tau^i)^{-1} \prod_{v \nmid p\infty} P_{\beta, v, \mathfrak{b}}(\chi_E(\pi_v)^{-1} |\pi_v|_v^{2s}),$$

where:

- (1) the product is over primes of E ;
- (2) the Hecke character χ_E is the restriction of χ to E ;
- (3) the function $P_{\beta, v, \mathfrak{b}}$ is a polynomial that is dependent only on β , v , and \mathfrak{b} and has coefficients in \mathbb{Z} and constant term 1;
- (4) the polynomial $P_{\beta, v, \mathfrak{b}}$ is identically 1 for all but finitely many v ;
- (5) τ is the Hecke character of E corresponding to K/E ;
- (6) π_v is a uniformizer of $\mathcal{O}_{E, v}$, viewed as an element of K^\times prime to p ;
- (7)

$$L^p(r, \chi_E^{-1} \tau^i) = \prod_{v \nmid p\infty \text{ cond } \tau} (1 - \chi_v(\pi_v)^{-1} \tau^i(\pi_v) |\pi_v|_v^r)^{-1}.$$

3.1.5. Global Fourier coefficients. Recall that by Lemma 4, the Fourier coefficients $c(\beta, h; f)$ are completely determined by the coefficients $c(\beta, 1_n; f)$. In Proposition 8, we combine the results of Sections 3.1.2, 3.1.3, and 3.1.4 in order to give the global Fourier coefficients of the Eisenstein series E_f .

Let χ be a unitary Hecke character as above, and furthermore, suppose the infinity type of χ is

$$(24) \quad \prod_{\sigma \in \Sigma} \sigma^{-k-2\nu(\sigma)} (\sigma \bar{\sigma})^{\frac{k}{2} + \nu(\sigma)}$$

(i.e. $k(\sigma) = k \in \mathbb{Z}$ for all $\sigma \in \Sigma$). Let $c(n, K)$ be the constant dependent only upon n and K defined in Equation (11).

Proposition 8. *Let $k \geq n$, let $\nu = (\nu(\sigma)) \in \mathbb{Z}^\Sigma$, and let*

$$(25) \quad f_{k, \nu, \chi, \tilde{F}} := f_{k, \nu, \chi, \mathfrak{b}, \tilde{F}} := \otimes_{v \in \Sigma} f_{\tilde{F}, v} \otimes f_{\infty}^{k, \nu}(\bullet; i1_n, \chi, s) \otimes f^{\mathfrak{b}} \in \text{Ind}_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)} (\chi \cdot |\cdot|_K^{-s}),$$

with χ as in Equation (24), $\otimes_{v \in \Sigma} f_{\tilde{F}, v}$ the section at p from Corollary 6, $f_{\infty}^{k, \nu}$ the section at ∞ defined in Section 3.1.2, and $f^{\mathfrak{b}}$ the section away from p and ∞ defined in Section 3.1.4.

Then at $s = \frac{k}{2}$, all the nonzero Fourier coefficients $c(\beta, 1_n; f_{k, \nu, \chi, \tilde{F}})$ are given by

$$(26) \quad D(n, K, \mathfrak{b}, p, k) \prod_{v \nmid p\infty} P_{\beta, v, \mathfrak{b}} (\chi_E(\pi_v)^{-1} |\pi_v|_v^k) \tilde{F}(1, {}^t\beta) \prod_{v \in \Sigma} \sigma_v(\det \beta)^{k-n} \mathbf{e}(i \text{tr}_{E/\mathbb{Q}}(\beta)).$$

where

$$\begin{aligned} D(n, K, \mathfrak{b}, p, k) &= C(n, K) N(\mathfrak{b} \mathcal{O}_E)^{-n^2} \prod_{i=0}^{n-1} \left(2^{(1-n)n_i - nk} (2\pi)^{nk} \left(\pi^{n(n-1)/2} \prod_{t=0}^{n-1} \Gamma(k-t) \right)^{-1} \right)^{[E:\mathbb{Q}]} \prod_{i=0}^{n-1} L^p(k-i, \chi_E^{-1} \tau^i)^{-1}. \end{aligned}$$

Proof. This follows directly from Equation (11), Corollary 6, and Equations (23) and (20). \blacksquare

Given \tilde{F} as above, define

$$\tilde{F}_\chi : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

to be the locally constant function whose support lies in

$$(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$$

and which is defined on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ by

$$(27) \quad \tilde{F}_\chi(x, y) = \prod_{v \in \Sigma} \chi_v(x) \tilde{F}(1, \mathbf{N}_{K/E}(x) {}^t y),$$

where the product is over the primes in Σ dividing p . Then for all $e \in \mathcal{O}_K^\times$,

$$\tilde{F}_\chi(ex, \mathbf{N}_{K/E}(e^{-1})y) = \mathbf{N}_{k, \nu}(e) \tilde{F}_\chi(x, y)$$

for all $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$ and $y \in M_{n \times n}(\mathcal{O}_E \times \mathbb{Z}_p)$. On the other hand, any locally constant function

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ which satisfies

$$F(ex, \mathbf{N}_{K/E}(e)^{-1}y) = \mathbf{N}_{E/\mathbb{Q}}(e)^k F(x, y)$$

for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n \times n}(\mathcal{O}_E \times \mathbb{Z}_p)$ can be written as a linear combination of such functions \tilde{F}_χ for Hecke characters χ of infinity type (k, ν) and conductor dividing p^∞ and functions \tilde{F} as above.

Now, let

$$G_{k, \nu, \chi, \tilde{F}} = D(n, K, \mathfrak{b}, p, k)^{-1} E_{f_{k, \nu, \chi, \tilde{F}}}.$$

Applying Proposition 8, we see that the Fourier coefficients of the holomorphic function $G_{k, \nu, \chi, \tilde{F}}(z, \frac{k}{2})$ on \mathcal{H}_n are all finite \mathbb{Z} -linear combinations (over a finite set of p -adic units $a \in K$) of terms of the form

$$(28) \quad \tilde{F}_\chi(a, \mathbf{N}_{K/E}(a)^{-1}\beta) \mathbf{N}_{k, \nu}(a^{-1} \det \beta) \mathbf{N}_{E/\mathbb{Q}}(\det \beta)^{-n}$$

(We remark that although π_v from Proposition 8 is a place of E for all v , the element a from Expression (28) might be in K but not \mathcal{O}_E , depending on our choice of cusp. The effect of the change of a cusp $m \in GM_+(\mathbb{A}_E)$, on q -expansions is given in Lemma 4.)

Thus, we obtain the following result.

Lemma 9. *Let $k \in \mathbb{Z}_{\geq n}$ and $\nu \in \mathbb{Z}^\Sigma$. For any locally constant function*

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ which satisfies

$$F(ex, \mathbf{N}_{K/E}(e)^{-1}y) = \mathbf{N}_{E/\mathbb{Q}}(e)^k F(x, y)$$

for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n \times n}(\mathcal{O}_E \times \mathbb{Z}_p)$, there is a C^∞ -automorphic form $G_{k, \nu, F}(z, s)$ of weight (k, ν) that is holomorphic at $s = k/2$ and whose Fourier coefficients at $s = k/2$ are \mathbb{Z} -linear combinations of terms of the form given in Expression (28).

(We obtain $G_{k, \nu, F}$ by taking a linear combination of the automorphic forms $G_{k, \nu, \chi, \tilde{F}}$.)

4. DIFFERENTIAL OPERATORS

4.1. C^∞ Differential Operators. In this section, we summarize results on C^∞ -differential operators from [Shi84a], [Shi84b], [Shi97], and [Shi00] that we need for the current paper; the reader can find many additional details about these operators in those four references (as well as numerous other papers by Shimura). Let $T = \mathrm{GL}_n(\mathbb{C})$; we identify T with the tangent space of \mathcal{H}_n . For each nonnegative integer d , let $\mathfrak{S}_d(T)$ denote the vector space of \mathbb{C} -valued homogeneous polynomial functions on T of degree d . (For instance,

the e -th power of the determinant function \det^e is in $\mathfrak{S}_{ne}(T)$.) We denote by τ the representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on $\mathfrak{S}_d(T)$ defined by

$$\tau(a, b)g(z) = g({}^tazb)$$

for all $a, b \in \mathrm{GL}_n(\mathbb{C})$, $z \in T$, and $g \in \mathfrak{S}_d$.

The classification of the irreducible subspaces of polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ and of irreducible subspaces of τ^r for each r is provided in [Shi84b, Section 2] and [Shi97, Sections 12.6 and 12.7]. We summarize the key features needed for our results; further details can be found in those two references. Given a matrix $a \in \mathrm{GL}_n(\mathbb{C})$, let $\det_j(a)$ denote the determinant of the upper left $j \times j$ submatrix of a . Each polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ can be composed into a direct sum of irreducible representations of $\mathrm{GL}_n(\mathbb{C})$. Each irreducible representation ρ of $\mathrm{GL}_n(\mathbb{C})$ contains a unique eigenvector p of highest weight $r_1 \geq \dots \geq r_n \geq 0$ (for a unique ordered n -tuple $r_1 \geq \dots \geq r_n \geq 0$ of integers dependent on ρ), which is a common eigenvector of the upper triangular matrices of $\mathrm{GL}_n(\mathbb{C})$ and satisfies

$$\begin{aligned} \rho(a)p &= \prod_{j=1}^n \det_j(a)^{e_j} p \\ (29) \quad e_j &= r_j - r_{j+1}, 1 \leq j \leq n-1 \end{aligned}$$

$$(30) \quad e_n = r_n,$$

for all a in the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. Also, for each ordered n -tuple $r_1 \geq \dots \geq r_n \geq 0$, there is a unique corresponding irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$. If ρ and σ are irreducible representations of $\mathrm{GL}_n(\mathbb{C})$, then by [Shi00, Theorem 12.7], $\rho \otimes \sigma$ occurs in τ^r if and only if the ρ and σ are representations of the same highest weights $r_1 \geq \dots \geq r_n$ as each other and $r_1 + \dots + r_n = r$. In this case, $\rho \otimes \sigma$ occurs with multiplicity one in τ^r , and the corresponding irreducible subspace of τ^r contains the polynomial $p(x) = \prod_{j=1}^n \det_j(x)^{e_j}$ (where e_j is defined as in Equations (29) and (30)); this polynomial $p(x)$ is an eigenvector of highest weight with respect to both ρ and σ .

Let (Z, τ_Z) be an irreducible subspace of (\mathfrak{S}_d, τ) of highest weight $r_1 \geq \dots \geq r_n$, and let $\zeta \in Z$. By [Shi84b], [Shi97, Section 23], and [Shi00, Section 13], there are C^∞ -differential operators $D_k(\zeta)$ that act on C^∞ -functions on \mathcal{H}_n and have the property that for all $\alpha \in U(\eta_n)$, $\zeta \in Z \subseteq S_d(T)$, and complex numbers s ,

$$(31) \quad D_k(\zeta) \left(\delta^s \|_{k, \nu} \alpha \right) = i^d \psi_Z(-k-s) \left(\delta^s \|_{k, \nu} \alpha \right) \cdot \zeta \left({}^t \eta^{-1t} \overline{\lambda_\alpha}^t \mu_\alpha^{-1} \right),$$

where (as proved in [Shi84b, Theorem 4.1])

$$\psi_Z(s) = \prod_{h=1}^n \prod_{j=1}^{r_h} (s - j + h).$$

Furthermore, as the proof of [Shi97, Lemma 23.4] explains, if ρ is the representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ defined by $\rho(a, b) = \det(b)^k$, then for all

C^∞ -functions f on \mathcal{H}_n ,

$$(32) \quad (D_k(\zeta)f) \|_{\rho \otimes \tau_Z} \alpha = D_k(\zeta) (f \|_{\rho} \alpha)$$

for all $\alpha \in G$. Note that when Z is a Σ -tuple $(Z_v)_{v \in \Sigma}$, we also use the notation ψ_Z to denote $\prod_{v \in \Sigma} \psi_{Z_v}$.

So for example, if $d \in \mathbb{Z}_{\geq 0}$ and $\zeta = \det^d$, then Equation (31) becomes

$$\begin{aligned} D_k(\det^d) (\delta^s \|_{k, \nu} \alpha) &= i^{nd} \psi_Z(-k-s) \delta^s \|_{k, \nu} \alpha \cdot \det^d ({}^t \eta^{-1} {}^t \overline{\lambda_\alpha} \cdot {}^t \mu_\alpha^{-1}) \\ &= \left(\frac{i}{2}\right)^{nd} \prod_{h=1}^n \prod_{j=1}^d (-k-s-j+h) \delta^{s-d} \|_{k+2d, \nu-d} \alpha. \end{aligned}$$

Consequently, if $d = (d_\sigma)_{\sigma \in \Sigma}$, then

$$\begin{aligned} &\left(\prod_{\sigma \in \Sigma} D_k(\det^{d(\sigma)}) \right) \left(G_{k, \nu, F} \left(z, \frac{k}{2} \right) \right) \\ &= \prod_{\sigma \in \Sigma} \left(\frac{i}{2} \right)^{nd(\sigma)} \prod_{h=1}^n \prod_{j=1}^{d(\sigma)} (-k-j+h) G_{k+2d, \nu-d, F} \left(z, \frac{k}{2} \right), \end{aligned}$$

as in [Eis11, Equation (43)].

As noted in [Shi84b, Section 6], $G_{k, \nu, F}(z, s)$ is a special case of the automorphic form $G_{k, \nu, \zeta, F}(z, s)$ defined similarly to $G_{k, \nu, F}(z, s)$, except that the section at ∞ is $([\delta^{s-\frac{k}{2}} \zeta (\eta^{-1})] \|_{\det^k \otimes \varphi} \alpha)(z)$, where $\zeta = (\zeta_v)_{v \in \Sigma}$ is a vector in an irreducible subrepresentation $(\varphi, Z) = \prod_{v \in \Sigma} (\varphi_v, Z_v)$ of τ^r in $\prod_{v \in \Sigma} S_{d_v}(T)$. By Equation (32), we have

$$D_k(\zeta) \left(G_{k, \nu, F} \left(z, \frac{k}{2} \right) \right) = \prod_{\sigma \in \Sigma} i^{d(\sigma)} \psi_{Z_v}(-k) G_{k, \nu, \zeta, F} \left(z, \frac{k}{2} \right),$$

where

$$D_k(\zeta) = \prod_{v \in \Sigma} D_k(\zeta_v).$$

The case where ζ is a highest weight vector will be of particular interest to us.

4.2. The Algebraic Geometric Setting. As explained in detail in [Eis12], which generalizes [Kat78], the C^∞ -differential operators discussed by Shimura have a geometric interpretation in terms of the Gauss-Manin connection. C^∞ -automorphic forms can, as explained in [Eis12], be interpreted as sections of a vector bundle on (the complex analytification of) the moduli spaces $\mathcal{M}_{n,n} = \text{Sh}(W)$. Applying a differential operator (as discussed in [Eis12]) to an automorphic form of weight ρ on $\mathcal{M}_{n,n}$ sends it to an automorphic form of weight $\rho \otimes \tau$ on $\mathcal{M}_{n,n}$. Fix an irreducible representation Z of $\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ (the complexification of the maximal compact $U(n) \times U(n)$ in $U(n, n)$, where $U(n)$ denotes a definite unitary group) contained in $\rho \otimes \tau$. Let ζ be a vector in Z .

In [Eis12, Section IX], we discussed a p -adic analogue $\theta(\zeta)$ of the differential operators $D_k(\zeta)$. The differential operators act on p -adic automorphic forms, viewed as sections of a vector bundle on the Igusa tower. The points of the Igusa tower parametrize tuples \underline{A} consisting of an ordinary abelian variety together with a polarization, endomorphism, and level structure. p -adic automorphic forms and the Igusa tower are discussed in detail in [Hid04, Chapter 8].

4.3. Rational Representations. In order to generalize our discussion from the C^∞ -setting to the p -adic setting, we briefly discuss rational representations and vector bundles, following [Hid04, Section 8.1.2] (which, in turn, summarizes relevant results from [Hid00] and [Jan87]). For a more detailed discussion of rational representations and vector bundles, especially in the context of p -adic automorphic forms, we recommend [Hid04, Section 8]. Also, [Hid04, Section 8] contains a more thorough discussion of p -adic automorphic forms. There is also a detailed discussion of the p -adic situation in [EHLS, Section 3].

Remark 10. A reader who is very familiar with the precise details of [Hid04, Section 8.1] might recall that the results in that section are for representations of GL_n and that, as Hida notes at the beginning of the section, he states his results only over \mathbb{Z} (and \mathbb{Q} , \mathbb{Q}_p , \mathbb{Z}_p , and $\mathbb{Z}/p^n\mathbb{Z}$) but notes that they can be extended to other rings. Because we are working with unitary groups, rather than symplectic groups, we must work with a CM field (and its integer ring), and we have made the corresponding necessary minor modifications (to which Hida alludes but does not state).

Let A be a ring. Let $B(A)$ denote the Borel subgroup of $\mathrm{GL}_n(\mathcal{O}_K \otimes_{\mathbb{Z}} A)$ consisting of upper triangular matrices in $\mathrm{GL}_n(\mathcal{O}_K \otimes A)$. Let $N(A)$ denote the unipotent subgroup of $B(A)$. Let $T(A) \cong B(A)/N(A)$ denote the torus. For each character κ of T , we denote by $R_{\mathcal{O}_K}[\kappa](A)$ the $A[\mathrm{GL}_n(\mathcal{O}_K \otimes A)]$ -module defined by

$$\begin{aligned} R_{\mathcal{O}_K}[\kappa](A) &= \{ \text{homogeneous polynomials } f : \mathrm{GL}_n(\mathcal{O}_K \otimes A)/N(A) \rightarrow A \mid \\ &\quad f(ht) = \kappa(t)f(h) \text{ for all } t \in T(A), h \in \mathrm{GL}_n(A)/N(A) \} \\ &= \mathrm{Ind}_{B(A)}^{\mathrm{GL}_n(\mathcal{O}_K \otimes A)}(\kappa). \end{aligned}$$

The group $\mathrm{GL}_n(\mathcal{O}_K \otimes A)/N(A)$ acts on $R_{\mathcal{O}_K}[\kappa](A)$ via

$$(g \cdot f)(x) = f(g^{-1}x).$$

We shall be particularly interested in the case in which $A = \prod_{\sigma \in \Sigma} \mathbb{C}$ and the case in which A is a p -adic ring. In these cases, we have (respectively)

$$\begin{aligned} \mathrm{GL}_n(\mathcal{O}_K \otimes A) &\cong \prod \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \\ \mathrm{GL}_n(\mathcal{O}_K \otimes A) &\cong \mathrm{GL}_n(\mathcal{O}_E \otimes A) \times \mathrm{GL}_n(\mathcal{O}_E \otimes A). \end{aligned}$$

In particular,

$$\mathrm{GL}_n(\mathcal{O}_K \otimes \mathbb{Z}_p) \cong \prod_{v \in \Sigma} \mathrm{GL}_n(\mathcal{O}_{E_v}) \times \mathrm{GL}_n(\mathcal{O}_{E_v}).$$

When A comes with a topology (in particular, if A is a p -adic ring), we denote by $\mathcal{C}_{\mathcal{O}_K}[\kappa](A)$ and $\mathcal{LC}_{\mathcal{O}_K}[\kappa](A)$ the modules

$$\begin{aligned} \mathcal{C}_{\mathcal{O}_K}[\kappa](A) &= \\ &\{ \text{continuous } f : \mathrm{GL}_n(\mathcal{O}_K \otimes A)/N(A) \rightarrow A \mid \\ &\quad f(ht) = \kappa(t)f(h) \text{ for all } t \in T(A), h \in \mathrm{GL}_n(A)/N(A) \} \\ \mathcal{LC}_{\mathcal{O}_K}[\kappa](A) &= \\ &\{ \text{locally constant } f : \mathrm{GL}_n(\mathcal{O}_K \otimes A)/N(A) \rightarrow A \mid \\ &\quad f(ht) = \kappa(t)f(h) \text{ for all } t \in T(A), h \in \mathrm{GL}_n(A)/N(A) \} \end{aligned}$$

on which $\mathrm{GL}_n(\mathcal{O}_K \otimes A)$ acts by

$$(g \cdot f)(x) = f(g^{-1}x).$$

So for each C^∞ -automorphic form f on $\prod_{v \in \Sigma} \mathcal{H}_n$ such that $f|_{k, \nu} \alpha = f$ (for all α in some congruence subgroup) and each highest weight vector ζ , we may view $D_k(\zeta)f$ as an $R_{\mathcal{O}_K}[\kappa_{k, \nu} \cdot \kappa](\mathbb{C})$ -valued function on \mathcal{H}_n , where $\kappa_{k, \nu}(t_1, \dots, t_n, t_{n+1}, \dots, t_{2n}) = \prod_{i=1}^n t_i^{k+\nu} t_{i+n}^{-\nu}$.

We now recall the setting of [Eis11, Section 3], as we will momentarily be in a similar (but not identical) situation. For any \mathcal{O}_K -algebra R , the R -valued points of ${}_{\kappa}\mathrm{Sh}(R)$ parametrize tuples \underline{A} consisting of an abelian variety together with a polarization, endomorphism, and level structure. We shall not need further details of these points here; the reader can find more details in a number of references, including [Lan08, Hid04, EHLS, Eis12]. Given a point \underline{A} in ${}_{\kappa}\mathrm{Sh}(R)$, we write $\underline{\omega}_{\underline{A}/R} = \underline{\omega}_{\underline{A}/R}^+ \oplus \underline{\omega}_{\underline{A}/R}^-$ for the sheaf of one-forms on \underline{A} . (As in [Eis12, Section 2], $\underline{\omega}_{\underline{A}/R}^+$ and $\underline{\omega}_{\underline{A}/R}^-$ are the rank n submodules determined by the action of \mathcal{O}_K .) We identify $G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/\mathcal{K}$ (which we identify with copies of \mathcal{H}_n) with the points of ${}_{\kappa}\mathrm{Sh}(\mathbb{C})$; we shall write $\underline{A}(z)$ to mean the point of \underline{A} identified with $z \in \prod_{v \in \Sigma} \mathcal{H}_n$ under this identification. Under this identification, if we fix an ordered basis of differentials u_1^\pm, \dots, u_n^\pm for $\underline{\omega}_{\underline{A}/\mathbb{C}}^\pm$, then an $R_{\mathcal{O}_K}[\kappa](\mathbb{C})$ -valued automorphic form f on \mathcal{H}_n corresponds to an automorphic form \tilde{f} on ${}_{\kappa}\mathrm{Sh}(\mathbb{C})$ via

$$f(z) = \tilde{f}(\underline{A}(z), u_1^\pm(z), \dots, u_n^\pm(z)),$$

(Each choice of an ordered basis gives an isomorphism of $\underline{\omega}_{\underline{A}/\mathbb{C}}^\kappa$ with $R_{\mathcal{O}_K}[\kappa](\mathbb{C})$.) Any other ordered basis of differentials for $\underline{\omega}_{\underline{A}/\mathbb{C}}^\pm$ is simply obtained by the linear action of $\mathrm{GL}_n(\mathcal{O}_K \otimes \mathbb{C}) \cong \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on $\underline{\omega}(z) = \underline{\omega}(z)^+ \oplus \underline{\omega}(z)^-$, and

$$\tilde{f}(\underline{A}(z), g \cdot (u_1^\pm(z), \dots, u_n^\pm(z))) = g \cdot (f(\underline{A}(z), u_1^\pm(z), \dots, u_n^\pm(z)))$$

As discussed in [Eis12], there are also p -adic differential operators $\theta(\zeta)$ that act on p -adic automorphic forms (viewed as sections of a vector bundle over a formal scheme over the ordinary locus of ${}_K\text{Sh}(R)$, for R a mixed characteristic discrete valuation ring with residue characteristic p) and take values in $R_{\mathcal{O}_K}[\kappa](R)$; for details on p -adic automorphic forms and the related geometry, see [Hid04, Section 8], [EHLS, Section 3], or [Eis12].

For any ring R , let eval_{1_n} denote the element of the dual module $(R_{\mathcal{O}_K}[\kappa](R))^{\vee} = \text{Hom}_R(R_{\mathcal{O}_K}[\kappa](R), R)$ defined by

$$(33) \quad \text{eval}_{1_n}(f) = f(1_n).$$

In [Eis12, Section 9], we gave a formula for the action of p -adic differential operators $\theta(\zeta)$ (with $\zeta = \prod_{v \in \Sigma} \zeta_v$, as above) on the q -expansions of p -adic automorphic forms. In particular, if f is a scalar-weight p -adic automorphic form whose q -expansion at given cusp $m \in GM$ is

$$f(q) = \sum_{\beta} a(\beta) q^{\beta},$$

then it follows from the formulas in [Eis12, Section 9] that

$$\text{eval}_{1_n}(\theta(\zeta)f)(q) = \sum_{\beta} a(\beta) \cdot \mathcal{F}_{\zeta}(\beta) q^{\beta},$$

where

$$(34) \quad \mathcal{F}_{\zeta} = \prod_{v \in \Sigma} \mathcal{F}_{\zeta_v}$$

with \mathcal{F}_{ζ_v} the function on $M_{n \times n}(R)$ (for any ring R) defined by

$$\begin{aligned} \mathcal{F}_{\zeta_v} : M_{n \times n}(R) &\rightarrow R \\ \beta &\mapsto \sum_{i=1}^l (g_i \cdot \zeta_v)(\beta), \end{aligned}$$

where $g_1 \cdot \zeta_v, \dots, g_l \cdot \zeta_v$ is a basis for $R_{\mathcal{O}_K}[\kappa](R)$.

4.4. CM Points and Pullbacks. In this section, we discuss the pullback of certain Eisenstein series to CM points. This material extends [Eis11, Section 3.0.1] beyond the case of scalar weights. Let R be an \mathcal{O}_K -subalgebra of $\overline{\mathbb{Q}} \cap \iota_{\infty}^{-1}(\mathcal{O}_{\mathbb{C}_p})$ in which p splits completely. Note that the embeddings ι_{∞} and ι_R restrict to R to give embeddings

$$\begin{aligned} \iota_{\infty} : R &\hookrightarrow \mathbb{C} \\ \iota_p : R &\hookrightarrow R_0 = \varprojlim_m R/p^m R. \end{aligned}$$

Let \underline{A} be a CM abelian variety with PEL structure over R (i.e. a CM point of the moduli space ${}_K\text{Sh}(R)$, or equivalently, a point of $Sh(U(n) \times U(n)) \rightarrow Sh(U(n, n))$). Note that by extending by scalars, we may also view \underline{A} as an abelian variety over \mathbb{C} or R_0 . As explained in [Kat78, Section 5.1] (for Hilbert modular forms) and [EHLS, Section 3.8.1] (for automorphic forms on $U(n, n)$), there are complex and p -adic periods $\Omega = (\Omega^+, \Omega^-) \in (\mathbb{C}^{\times})^n \times (\mathbb{C}^{\times})^n$

and $c = (\mathcal{O}_{\mathbb{C}_p}^\times)^n \times (\mathcal{O}_{\mathbb{C}_p}^\times)^n$ attached to each CM abelian variety \underline{A} over R ; applying the main results on algebraicity from [Eis12], we see that

$$(35) \quad \begin{aligned} & (\kappa \cdot \kappa_{k,\nu})^{-1}(\Omega) \prod_{\sigma \in \Sigma} \kappa_\sigma(2\pi i) \psi_Z(-k) G_{k,\nu,\zeta,F} \left(z; h, \chi, \mu, \frac{k}{2} \right) \\ &= (\kappa \cdot \kappa_{k,\nu})^{-1}(c) \theta(\zeta) G_{k,\nu,F}(\underline{A}), \end{aligned}$$

where z is a point in $\prod_{\sigma \in \Sigma} \mathcal{H}_n$ corresponding to the CM abelian variety \underline{A} viewed as an abelian variety over \mathbb{C} (by extending scalars to \mathbb{C}). Here, Z is the irreducible subrepresentation of $\prod_{v \in \Sigma} \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ of highest weight $\kappa \in (\mathbb{Z}^n)^\Sigma$ and has ζ as a highest weight vector; by $\kappa(a)$ with a a scalar, we mean κ evaluated at the n -tuple (a, \dots, a) in the torus. (The periods Ω and c can be defined uniformly for all CM points at once, as explained in [Kat78, Section 5.1] and [EHLS, Section 3.8.1]. For the present paper, though, this is not necessary.)

Remark 11. We may view the pullback of an automorphic form on $U(n, n)$ to $U(n) \times U(n)$ as an automorphic form on this smaller space. Since the maximal compact at ∞ in $U(n, n)$ is $U(n) \times U(n)$ (i.e. the same as the maximal compact at ∞ in $U(n) \times U(n)$), a p -adic family of automorphic forms on $U(n, n)$, parametrized by weight, automatically pulls back to a p -adic family of automorphic forms on $U(n) \times U(n)$ parametrized by these same weights. In [Eis], we explain how to construct p -adic families of automorphic forms on $U(n, n)$ that pull back to families on non-definite unitary groups that are parametrized by the weights on these non-definite unitary groups; note that in the non-definite case, a highest weight vector for the maximal compact in $U(n, n)$ is not necessarily a highest weight vector for the maximal compact in $U(a, b) \times U(b, a)$ for $a + b = n$ with $ab \neq 0$.

5. A p -ADIC EISENSTEIN MEASURE WITH VALUES IN THE SPACE OF VECTOR-WEIGHT AUTOMORPHIC FORMS

5.1. p -adic Eisenstein Series. As we explain in Theorem 12, when R is a (profinite) p -adic ring, we can extend Theorem 2 to the case of continuous (not necessarily locally constant) functions F .

Theorem 12. *Let R be a (profinite) p -adic \mathcal{O}_K -algebra. Fix an integer $k \geq n$, and let $\nu = (\nu(\sigma))_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$. Let*

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

be a continuous function supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ which satisfies

$$F(ex, \mathbf{N}_{K/E}(e)^{-1}y) = \mathbf{N}_{k,\nu}(e) F(x, y)$$

for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$. Then there exists a p -adic automorphic form $G_{k,\nu,F}$ whose q -expansion at a cusp $m \in \mathrm{GM}$ is of

the form $\sum_{0 < \beta \in L_m} c(\beta) q^\beta$ (where L_m is the lattice in $\text{Her}_n(K)$ determined by m), with $c(\beta)$ a finite \mathbb{Z} -linear combination of terms of the form

$$F(a, \mathbf{N}_{K/E}(a)^{-1} \beta) \mathbf{N}_{k,\nu}(a^{-1} \det \beta) \mathbf{N}_{E/\mathbb{Q}}(\det \beta)^{-n}$$

(where the linear combination is the sum over a finite set of p -adic units $a \in K$ dependent upon β and the choice of cusp $m \in GM$).

Proof. The proof is similar to the the proof of [Kat78, Theorem (3.4.1)]. We remind the reader of the idea of [Kat78, Theorem (3.4.1)]. For each integer $j \geq 1$, define

$$\begin{aligned} F_j : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p) &\rightarrow R/p^j R \\ F_j(x, y) &= F(x, y) \pmod{p^j R}. \end{aligned}$$

Then F_j is a locally constant function satisfying the conditions of Theorem 2. So by the q -expansion principle for p -adic forms ([Hid05, Corollary 10.4], [Hid04, Section 8.4]), there is a p -adic automorphic form $G_{k,\nu,F}$ whose q -expansion satisfies the conditions in the statement of the theorem. \blacksquare

Corollary 13. *Let R be a (profinite) p -adic \mathcal{O}_K -algebra, let $\nu = (\nu(\sigma))_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$, and let $k \geq n$ be an integer. Let*

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

be a continuous function supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ which satisfies

$$F(ex, \mathbf{N}_{K/E}(e)^{-1}y) = \mathbf{N}_{k,\nu}(e) F(x, y)$$

for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$. Then

$$G_{k,\nu,F} = G_{n,0,\mathbf{N}_{k-n,\nu}}(x^{-1} \mathbf{N}_{K/E}(x)^n \det y) F(x, y),$$

where

$$\mathbf{N}_{k-n,\nu}(x^{-1} \mathbf{N}_{K/E}(x)^n \det y) F(x, y),$$

denotes the function defined by

$$(x, y) \mapsto \mathbf{N}_{k-n,\nu}(x^{-1} \mathbf{N}_{K/E}(x)^n \det y) F(x, y).$$

on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$ and extended by 0 to all of $(\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$.

Proof. This follows from the q -expansion principle [Hid05, Corollary 10.4]. \blacksquare

5.2. Eisenstein Measures. In analogue with [Kat78, Lemma (4.2.0)] (which handles the case of Hilbert modular forms), we have the following lemma (which applies to all integers $n \geq 1$).

Lemma 14. *Let R be a p -adic \mathcal{O}_K -algebra. Then the inverse constructions*

$$(36) \quad H(x, y) = \frac{1}{\mathbf{N}_{n,0}(x\mathbf{N}_{K/E}(x)^{-n} \det y)} F(x, y^{-1})$$

$$(37) \quad F(x, y) = \frac{1}{\mathbf{N}_{n,0}(x^{-1}\mathbf{N}_{K/E}(x)^n \det y)} H(x, y^{-1})$$

give an R -linear bijection between the set of continuous R -valued functions

$$F : (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

satisfying

$$F(ex, \mathbf{N}_{K/E}(e)^{-1}y) = \mathbf{N}_{n,0}(e) F(x, y)$$

for all $e \in \mathcal{O}_K^\times$ and the set of continuous R -valued functions

$$H : (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

satisfying

$$H(ex, \mathbf{N}_{K/E}(e)y) = H(x, y)$$

for all $e \in \mathcal{O}_K^\times$.

Proof. The proof follows immediately from the properties of F and H . ■

Let

$$(38) \quad \mathcal{G}_n = ((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)) / \overline{\mathcal{O}_K^\times},$$

where $\overline{\mathcal{O}_K^\times}$ denotes the p -adic closure of \mathcal{O}_K^\times embedded diagonally (as $(e, \mathbf{N}_{K/E}(e))$) in $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$ (and as before, $(\mathcal{O}_E \otimes \mathbb{Z}_p)^\times$ is embedded diagonally inside of $\mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$). Then Lemma 14 gives a bijection between the R -valued continuous functions H on \mathcal{G}_n and the R -valued continuous functions F on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$ satisfying $F(ex, \mathbf{N}_{K/E}(e)^{-1}y) = \mathbf{N}_{n,0}(e) F(x, y)$ for all $e \in \mathcal{O}_K^\times$.

The topic of p -adic measures is discussed in detail in [Was97] and summarized in [Kat78, Section 4.0]. In brief, for any (profinite) p -adic ring R , an R -valued p -adic measure on a (profinite) compact, totally disconnected topological space X is a \mathbb{Z}_p -linear map

$$\mu : \mathcal{C}(Y, \mathbb{Z}_p) \rightarrow R,$$

or equivalently (as explained in [Kat78, Section 4.0]), an R' -linear map

$$\mu : \mathcal{C}(Y, R') \rightarrow R$$

for any p -adic ring R' such that R is an R' -algebra. Instead of $\mu(f)$, one typically writes

$$\int_Y f d\mu.$$

In Theorem 15, we specialize to the case where R is the ring $\mathcal{V}_{n,n}$ of p -adic automorphic forms on $U(n,n)$ and Y is the group \mathcal{G}_n defined in Equation (38).

Theorem 15 (A p -adic Eisenstein Measure for Vector-Weight Automorphic Forms). *Let R be a profinite p -adic ring. There is a $\mathcal{V}_{n,n}$ -valued p -adic measure $\mu = \mu_{\mathfrak{b},n}$ on \mathcal{G}_n defined by*

$$\int_{\mathcal{G}_n} H d\mu_{\mathfrak{b},n} = G_{n,0,F}$$

for all continuous R -valued functions H on \mathcal{G}_n , with

$$F(x, y) = \frac{1}{\mathbf{N}_{n,0}(x^{-1}\mathbf{N}_{K/E}(x)^n \det y)} H(x, y^{-1})$$

extended by 0 to all of $(\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$.

Proof. Note that F is the function corresponding to H under the bijection in Lemma 14. The theorem then follows immediately from Theorem 12, Corollary 13, Lemma 14, and the q -expansion principle. \blacksquare

Note that the measure $\mu_{\mathfrak{b},n}$ depends only upon n and \mathfrak{b} . In Section 6, we relate the measure $\mu_{\mathfrak{b},n}$ to the Eisenstein measure in [Kat78] and comment on how $\mu_{\mathfrak{b},n}$ can be modified to the case of Siegel modular forms (i.e. automorphic forms on symplectic groups).

Let \mathcal{F}_ζ be defined as in Equation (34), and let eval_{1_n} be defined as in Equation (33). It follows from the definition of the measure $\mu_{\mathfrak{b},n}$ in Theorem 15 that

$$\int_{\mathcal{G}_n} H(x, y) \mathcal{F}_\zeta(\mathbf{N}_{K/E}(x) y^{-1}) d\mu_{\mathfrak{b},n} = \text{eval}_{1_n}(\theta(\zeta) G_{n,0,F(x,y)}),$$

and

$$\begin{aligned} \int_{\mathcal{G}_n} H(x, y) \mathcal{F}_{\det^d}(\mathbf{N}_{K/E}(x) y^{-1}) d\mu_{\mathfrak{b},n} \\ = \int_{\mathcal{G}_n} H(x, y) \det(\mathbf{N}_{K/E}(x)^{-1} y)^{-d} d\mu_{\mathfrak{b},n} \\ = G_{n+2d, -2d, F(x,y)}. \end{aligned}$$

Now, let \underline{A} be a CM abelian variety with PEL structure over a subring R of $\overline{\mathbb{Q}} \cap \mathcal{O}_{\mathbb{C}_p}$ (i.e. a CM point of the moduli space ${}_K\text{Sh}(R)$, or equivalently, a point of $Sh(U(n) \times U(n)) \hookrightarrow Sh(U(n, n))$). As discussed above, by extending by scalars, we may also view \underline{A} as an abelian variety over \mathbb{C} or over $R_0 = \varprojlim_m R/p^m R$. It follows from Equation (35) that

(39)

$$\begin{aligned} (\kappa \cdot \kappa_{k,\nu})^{-1}(c) \left(\int_{\mathcal{G}_n} H(x, y) \mathcal{F}_\zeta(\mathbf{N}_{K/E}(x) y^{-1}) d\mu_{\mathfrak{b},n} \right) (\underline{A}) \\ = (\kappa \cdot \kappa_{k,\nu})^{-1}(\Omega) \prod_{\sigma \in \Sigma} \kappa_\sigma(2\pi i) \psi_Z(-k) (\text{eval}_{1_n}(G_{k,\nu,\zeta,F})) \left(z, \frac{k}{2} \right), \end{aligned}$$

where z is a point in $\prod_{\sigma \in \Sigma} \mathcal{H}_n$ corresponding to the CM abelian variety \underline{A} viewed as an abelian variety over \mathbb{C} (by extending scalars to \mathbb{C}) and Ω and c are the periods from Equation (35). Here, Z is the irreducible subrepresentation of $\prod_{\sigma \in \Sigma} \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ of highest weight κ and has ζ as a highest weight vector; by $\kappa(a)$ with a a scalar, we mean κ evaluated at the n -tuple (a, \dots, a) in the torus.

In other words, the p -adic measure $\mu_{\mathfrak{b},n}$ allows us to p -adically interpolate the values of the C^∞ - (not necessarily holomorphic) function $G_{k,\nu,\zeta,F}(z, \frac{k}{2})$ at CM points z .

Theorem 16. *For each abelian variety \underline{A} defined over a (profinite) p -adic \mathcal{O}_K -algebra R_0 , there is an R_0 -valued p -adic measure $\mu(\underline{A}) := \mu_{\mathfrak{b},n}(\underline{A})$ defined by*

$$\int_{\mathcal{G}_n} H d\mu_{\mathfrak{b},n}(\underline{A}) = G_{n,0,F}(\underline{A});$$

and when $R_0 = \varprojlim_m R/p^m R$ with $R \subseteq \bar{\mathbb{Q}}$ and \underline{A} is a CM point defined over R ,

$$\begin{aligned} & (\kappa \cdot \kappa_{k,\nu})^{-1}(c) \int_{\mathcal{G}_n} H(x, y) \mathcal{F}_\zeta(\mathbf{N}_{K/E}(x) y^{-1}) d\mu_{\mathfrak{b},n}(\underline{A}) \\ &= (\kappa \cdot \kappa_{k,\nu})^{-1}(\Omega) \prod_{\sigma \in \Sigma} \kappa_\sigma(2\pi i) \psi_Z(-k) (\mathrm{eval}_{1_n}(G_{k,\nu,\zeta,F})) \left(z, \frac{k}{2}\right) \end{aligned}$$

with $z \in \prod_{v \in \Sigma} \mathcal{H}_n$ corresponding to the CM abelian variety \underline{A} viewed as an abelian variety over \mathbb{C} .

As noted in Remark 11, the pullback of an automorphic form on $U(n, n)$ to $U(n) \times U(n)$ is automatically an automorphic form on the product of definite unitary groups $U(n) \times U(n)$. So Theorem 15 also gives a p -adic measure with values in the space of automorphic forms on the product of definite unitary groups $U(n) \times U(n)$. In [Eis], we explain how to modify our construction to obtain p -adic measures with values in the space of automorphic forms on non-definite groups.

Remark 17 (Relationship to the Eisenstein Measures in [Eis11]). The measure $\mu_{\mathfrak{b},n}$ can be pulled back to measures similar to the ones in [Kat78, Eis11, EHLS], but unlike those measures, $\mu_{\mathfrak{b},n}$ handles both scalar and vector weights at once and does not require a choice of a partition of n . It is “universal” in the sense that it encompasses all signatures (important in [Eis, Eis11]) and partitions of n , as well as all weights, at once.

For the curious reader, although we shall not need this remark anywhere else in this paper, we briefly explain the relationship between the measure $\mu_{\mathfrak{b},n}$ defined in Theorem 15 and the measure ϕ defined in [Eis11, Theorem 20]. For each $v \in \Sigma$, let $r_v = r(v)$ be a positive integer $\leq n$, and let $r = (r_v)_v \in$

\mathbb{Z}^Σ . As in [Eis11, Equation (33)], let

$$(40) \quad T(r) = \prod_{v \in \Sigma} \underbrace{\mathcal{O}_{E_v}^\times \times \cdots \times \mathcal{O}_{E_v}^\times}_{r_v \text{ copies}}.$$

Let $\rho = \prod_{v \in \Sigma} (\rho_{1,v}, \dots, \rho_{r(v),v})$ be a p -adic character on $T(r)$, let $n = n_{1,v} + \cdots + n_{r(v),v}$ be a partition of n for each $v \in \Sigma$, and let F_ρ be the function on $M_{n \times n}(E)$ defined by

$$F_\rho(x) := \prod_{v \in \Sigma} \prod_{i=1}^{r(v)} \rho_{i,v}(\det_{n_i}(x)),$$

with \det_j defined as on page 22. Let χ be a p -adic function supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \overline{\mathcal{O}_K}^\times$ and extended by 0 to all of $\mathcal{O}_K \otimes \mathbb{Z}_p$. Let $H_{\rho,\chi}$ be the function corresponding via the bijection in Lemma 14 to the function $F_{\rho,\chi}$ supported on \mathcal{G}_n (and extended by 0) defined by

$$F_{\rho,\chi}(x, y) = \chi(x) \mathbb{N}_{n,0}(x) F_\rho(\mathbf{N}_{K/E}(x)^t y).$$

Then

$$\int_{\mathcal{G}_n} H_{\rho,\chi} d\mu_{\mathbf{b},n} = \int_{(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \overline{\mathcal{O}_K}^\times \times T(r)} (\chi, \rho) d\phi.$$

Note that the measure ϕ is dependent upon the choice of r and the choice of the partition of n , while the measure $\mu_{\mathbf{b},n}$ is independent of both of these choices.

6. REMARKS ABOUT THE CASE OF SYMPLECTIC GROUPS, SIEGEL MODULAR FORMS, AND KATZ'S EISENSTEIN MEASURE FOR HILBERT MODULAR FORMS

Note that the case of Siegel modular forms is quite similar. We essentially just need to replace the CM field K with the totally real field E throughout. Once we have replaced K by E , $\mathbf{N}_{k,\nu}$ becomes $\mathbf{N}_{E/\mathbb{Q}}^k$, and $\mathbf{N}_{K/E}$ becomes the identity map. Consequently, Equations (36) and (37) become

$$\begin{aligned} H(x, y) &= \frac{1}{\mathbf{N}_{E/\mathbb{Q}}(x^{1-n} \det y)^n} F(x, y^{-1}) \\ F(x, y) &= \frac{1}{\mathbf{N}_{E/\mathbb{Q}}(x^{-1+n} \det y)^n} H(x, y^{-1}) \end{aligned}$$

To highlight the similarity with [Kat78], we note that when $n = 1$, these equations become

$$\begin{aligned} H(x, y) &= \frac{1}{\mathbf{N}_{E/\mathbb{Q}}(y)} F(x, y^{-1}) \\ F(x, y) &= \frac{1}{\mathbf{N}_{E/\mathbb{Q}}(y)} H(x, y^{-1}) \end{aligned}$$

This relationship between H and F is similar to the relationship between the functions denoted H and F by Katz in [Kat78], which play a similar role to

the functions we denoted by H and F . (The minor difference between Katz's relationship between H and F and ours is due the fact that throughout the paper, his $F(x, y)$ is our $F(y, x)$, i.e. our first variable plays the role of his second variable and vice versa throughout the paper.)

The differential operators are developed from the C^∞ -perspective simultaneously for both unitary and symplectic groups in [Shi00]. As noted on [Eis12, p. 4], in [Eis12, Section 3.1.1], and in [Pan05, CP04], the algebraic geometric and p -adic formulation of the operators for Siegel modular forms (i.e. for symplectic groups) is similar. In the case of Siegel modular forms, the algebraic geometric formulation of the differential operators is discussed in [Har81]. Also, the case of symplectic groups is handled directly alongside the case of unitary groups in Hida's discussion of p -adic automorphic forms in [Hid04]. So the construction in this paper carries over with only minor changes (essentially, replacing K by E throughout) to the case of symplectic groups over a totally real field E and automorphic forms (Siegel modular forms) on those groups.

6.1. The case $n = 1$. Continuing with the symplectic case with $n = 1$, Theorem 2 becomes

Theorem 18. *Let R be an \mathcal{O}_E -algebra, let and let $k \geq 1$ be an integer. For each locally constant function*

$$F : (\mathcal{O}_E \otimes \mathbb{Z}_p) \times (\mathcal{O}_E \otimes \mathbb{Z}_p) \rightarrow R$$

supported on $(\mathcal{O}_E \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_E \otimes \mathbb{Z}_p)^\times$ which satisfies

$$(41) \quad F(ex, e^{-1}y) = \mathbf{N}_{E/\mathbb{Q}}(e)^k F(x, y)$$

for all $e \in \mathcal{O}_E^\times$, $x \in \mathcal{O}_E \otimes \mathbb{Z}_p$, and $y \in \mathcal{O}_E \otimes \mathbb{Z}_p$, there is a Hilbert modular form $G_{k,F}$ of weight k defined over R whose q -expansion at a cusp $m \in GM$ is of the form $\sum_{\beta > 0} c(\beta)q^\beta$ (where L_m is the lattice in E determined by m), with $c(\beta)$ a finite \mathbb{Z} -linear combination of terms of the form

$$F(a, (a)^{-1}\beta) \mathbf{N}(a^{-1}\beta)^k \mathbf{N}_{E/\mathbb{Q}}(\beta)^{-1}$$

(where the linear combination is a sum over a finite set of p -integral $a \in E$ dependent upon β and the choice of cusp $m \in GM$).

Still continuing with the symplectic case with $n = 1$, Theorem 15 becomes

Theorem 19. *There is a measure μ on*

$$\mathcal{G} = ((\mathcal{O}_E \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_E \otimes \mathbb{Z}_p)^\times) / \overline{\mathcal{O}_E^\times}$$

(with values in the space of p -adic Hilbert modular forms) defined by

$$\int_{\mathcal{G}} H d\mu = G_{1,F}$$

for all continuous R -valued functions H on \mathcal{G} , with

$$F(x, y) = \frac{1}{\mathbf{N}_{E/\mathbb{Q}}(y)} H(x, y^{-1})$$

extended by 0 to all of $(\mathcal{O}_E \otimes \mathbb{Z}_p) \times (\mathcal{O}_E \otimes \mathbb{Z}_p)$.

Note that we have essentially recovered the Eisenstein series and measure from [Kat78]. (Again, the difference between Katz's order of the variables x and y and ours is due to the fact that throughout the paper, his $F(x, y)$ is our $F(y, x)$, i.e. our first variable plays the role of his second variable and vice versa throughout the paper.) The reader familiar with [Kat78] will notice the similarities with [Kat78, (5.5.1)-(5.5.7)]. In particular, let χ be a Grössencharacter of the CM field K whose conductor divides p^∞ and whose infinity type is

$$-k \sum_{\sigma \in \Sigma} \sigma - \sum_{\sigma \in \Sigma} d(\sigma) (\sigma - \bar{\sigma})$$

with $d(\sigma) \geq 0$ for all $\sigma \in \Sigma$ and $k \geq n$. We view χ as an $\mathcal{O}_{\mathbb{C}_p}$ -valued character on $\mathbb{A}^{\infty, \times} \times \prod_{v \in \Sigma} \overline{\mathbb{Q}}$ and consider its restriction to the subring consisting of elements $((1_v)_{v \nmid p^\infty}, a, a)$, with $a \in \mathcal{O}_K \otimes \mathbb{Z}_{(p)}$, which is a subring of

$$(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \xrightarrow{\sim} (\mathcal{O}_E \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_E \otimes \mathbb{Z}_p)^\times.$$

Then we have

$$\begin{aligned} \chi(\alpha) &= \chi_{\text{finite}}(\alpha) \cdot \frac{\prod_{\sigma \in \Sigma} \sigma(\bar{\alpha})^{d(\sigma)}}{\prod_{\sigma \in \Sigma} \sigma(\alpha)^{k+d(\sigma)}} \\ \chi(x, y) &= \chi_{\text{finite}}(x, y) \cdot \frac{\prod_{\sigma \in \Sigma} \sigma(y)^{d(\sigma)}}{\prod_{\sigma \in \Sigma} \sigma(x)^{k+d(\sigma)}}, \end{aligned}$$

with χ_{finite} a locally constant function. If

$$\begin{aligned} F(x, y) &= \frac{1}{\mathbf{N}(y)} \chi\left(x, \frac{1}{y}\right), \\ &= \chi_{\text{finite}}\left(x, \frac{1}{y}\right) \cdot \mathbf{N}(y)^{k-1} \prod_{\sigma \in \Sigma} \sigma(xy)^{d(\sigma)}, \end{aligned}$$

then

$$(42) \quad \int_{\mathcal{G}} \chi(x, y) d\mu_{\mathbf{b}, 1} = G_{1, F}$$

$$(43) \quad = G_{1, \chi_{\text{finite}}\left(x, \frac{1}{y}\right) \mathbf{N}(y)^{k-1} \prod_{\sigma \in \Sigma} \sigma(xy)^{d(\sigma)}}$$

$$(44) \quad = G_{k, \chi_{\text{finite}}\left(x, \frac{1}{y}\right) \prod_{\sigma \in \Sigma} \sigma(xy)^{d(\sigma)}}$$

$$(45) \quad = \left(\prod_{\sigma \in \Sigma} \theta(\sigma)^{d(\sigma)} \right) \left(G_{k, \chi_{\text{finite}}\left(x, \frac{1}{y}\right)} \right),$$

where $\theta(\sigma)$ denotes the $(\sigma$ -component of the) differential operator $\theta(\det)$ acting on automorphic forms in the 1-dimensional, symplectic case. Note the similarity of Equations (42) through (45) with [Kat78, Equations (5.5.6)-(5.5.7)].

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